Standing Wave Variance Budget

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Here we derive a form of the tracer variance budget for standing waves. This reveals how the transient eddies contribute to maintaining the standing wave.

The model potential temperature equation is

$$\theta_t + \boldsymbol{u} \cdot \nabla\theta = \kappa_h \nabla_h^2 \theta + (\kappa_v \theta_z)_z - \frac{\lambda}{\delta}(\theta_0 - \theta^*) \ . \tag{1}$$

Here κ_h is a spatially uniform horizontal diffusivity and κ_v is a vertical diffusivity that is elevated in the mixed layer (due to the KPP scheme). The final term represents the surface forcing, active only in the top model level; λ is a temperature relaxation inverse timescale, and δ is the thickness of the top grid cell. The surface temperature θ_0, and the relaxation profile $\theta^* = \theta^*(y)$.

We define two forms of average: time average and zonal average. The time average of a variable $A(x, y, z, t)$ is

$$\overline{A}(x, y, z) = \Delta T^{-1} \int_t^{t+\Delta T} A dt \ . \tag{2}$$

We will only take zonal averages of the time-averaged fields. Other conventions are possible (Viebahn and Eden, 2012), but this is the most informative decomposition. The zonal average is

$$\langle A \rangle (y, z) = \langle \overline{A} \rangle = L_x^{-1} \int_0^{L_x} \overline{A} dx \ . \tag{3}$$

We define the anomalies as follows:

$$A' = A - \overline{A} \tag{4}$$

$$A^\dagger = \overline{A} - \langle A \rangle \tag{5}$$

such that $A = \langle A \rangle (y, z) + A^\dagger (x, y, z) + A'(x, y, z, t)$. The standing wave is associated with $A^\dagger$.

The time average of (1), assuming a statistically steady state, is

$$\overline{\boldsymbol{u}} \cdot \nabla\overline{\theta} + \nabla \cdot \overline{\boldsymbol{u}'\theta'} = \kappa_h \nabla_h^2 \overline{\theta} + (\overline{\kappa_v \theta_z})_z - \frac{\lambda}{\delta}(\overline{\theta}_0 - \theta^*) \tag{6}$$

which can be expanded in terms of zonal mean and standing components as

$$\langle\boldsymbol{u}\rangle\cdot\nabla\langle\theta\rangle+\langle\boldsymbol{u}\rangle\cdot\nabla\theta^\dagger+\boldsymbol{u}^\dagger\cdot\nabla\langle\theta\rangle+\nabla\cdot\langle\boldsymbol{u}^\dagger\theta^\dagger\rangle+\nabla\cdot\overline{\boldsymbol{u}'\theta'} = \kappa_h\nabla_h^2\langle\theta\rangle+\kappa_h\nabla_h^2\theta^\dagger+(\langle\kappa_v\theta_z\rangle)_z+(\kappa_v\theta_z)_z^\dagger-\frac{\lambda}{\delta}(\langle\theta\rangle_0+\theta_0^\dagger-\theta \tag{7}$$

Finally, we can take a zonal time average of this equation to get the time- and zonally averaged $\langle\theta\rangle$ equation:

$$\langle\boldsymbol{u}\rangle \cdot \nabla\langle\theta\rangle + \nabla \cdot \langle\boldsymbol{u}^\dagger\theta^\dagger\rangle + \nabla \cdot \langle\overline{\boldsymbol{u}'\theta'}\rangle = \kappa_h \nabla_h^2 \langle\theta\rangle + (\langle\kappa_v\theta_z\rangle)_z - \frac{\lambda}{\delta}(\langle\theta\rangle_0 - \theta^*) \ . \tag{8}$$

First we construct the variance budget for $\overline{\theta'}/2$ in the standard way by subtracting (6) from (1), multiplying by θ' and taking a time average. The result is

$$\nabla \cdot \left(\overline{\boldsymbol{u}}\frac{\overline{\theta'^2}}{2} + \frac{\overline{\boldsymbol{u}'\theta'^2}}{2} \right) + \overline{\boldsymbol{u}'\theta'} \cdot \nabla\overline{\theta} \simeq -\kappa_h\overline{|\nabla\theta'|^2} - \overline{\kappa_v|\theta'_z|^2} - \frac{\lambda}{\delta}\overline{\theta_0'^2} \ , \tag{9}$$

where we have assumed a statistically steady state. The equation is not exact; for compactness we have neglected terms due to the diffusion of variance, which are demonstrably tiny in our context. By taking a global integral of this equation, we see the familiar result that, for steady states, a down-gradient eddy flux across the mean θ gradient (the last term on the LHS) is sustained by one of the dissipative processes on the RHS (Marshall and Shutts, 1981; Rhines and Young, 1982). Locally, the flux can be up or down gradient, due to the variance advection terms (Wilson and Williams, 2004).

A budget for the standing wave variance can be derived in an analogous way, by subtracting (8) from (7), multiplying by $\theta^\dagger$ and taking a zonal average. We find

$$\nabla\cdot\left(\langle\boldsymbol{u}\rangle\frac{\langle\theta^{\dagger 2}\rangle}{2}+\frac{\langle\boldsymbol{u}^\dagger\theta^{\dagger 2}\rangle}{2}\right)+\langle\boldsymbol{u}^\dagger\theta^\dagger\rangle\cdot\nabla\langle\theta\rangle+\langle\theta^\dagger\nabla\cdot(\overline{\boldsymbol{u}'\theta'})\rangle \simeq -\kappa_h\langle|\nabla\theta^\dagger|^2\rangle-\langle\kappa_v|\theta_z^\dagger|^2\rangle-\frac{\lambda}{\delta}\langle\theta_0^{\dagger 2}\rangle \ . \tag{10}$$

All the terms on the LHS of (10) are plotted in Fig. 1. The dissipation terms on the RHS side were not calculated, but they are clearly evident in the residual at the surface. Analysis of our simulations shows that the dominant balance in the budget is

$$\langle v^\dagger\theta^\dagger\rangle\langle\theta\rangle_y + \langle\theta^\dagger\nabla_h \cdot (\overline{\boldsymbol{u}'\theta'})\rangle \simeq 0 \ . \tag{11}$$

In words, this states that the standing eddy meridional heat flux is sustained not by irreversible mixing processes (as in (9)) but rather by a new term, the correlation between $\theta^\dagger$ and the convergence of lateral transient eddy fluxes. This approximate balance is illustrated in Fig. 2.

If we wish to develop a quasi-linear analytical model of the standing wave, we must find a way to represent the transient eddy term in (10), which is clearly crucial for the standing wave heat transport. We propose to represent it simply as a linear damping of the standing-wave, i.e.

$$\nabla_h \cdot (\overline{\boldsymbol{u}'\theta'}) \simeq \gamma\theta^\dagger \tag{12}$$

With a value of $\gamma = 10^{-7}$ s-1, this simple approximation qualitatively reproduces the structure and magnitude of the transient eddy flux divergence, as illustrated in Fig. 3.

References

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Rhines, P. B. and W. R. Young, 1982: Homogenization of potential vorticity in planetary gyres. *J. Fluid Mech.*, **122**, 347–367.

Viebahn, J. and C. Eden, 2012: Standing eddies in the meridional overturning circulation. *J. Phys. Oceanogr.*, **42**, 1486–1508.

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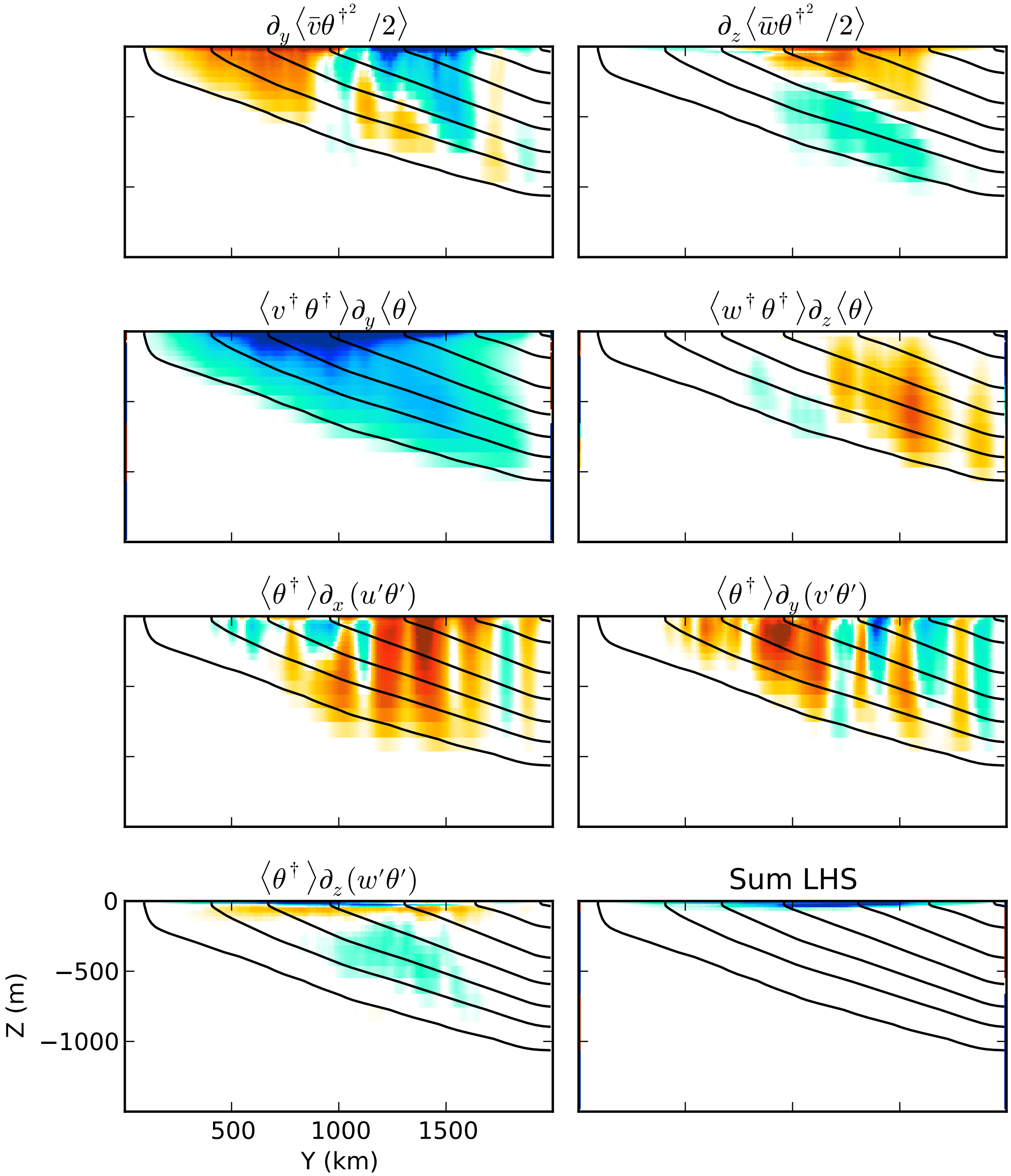

Figure 1: The terms in the variance budget (10). The magnitude of the color scale is $5{\times}10^{-8}$ $(^\circ\ \mathrm{C})^2\ \mathrm{s}^{-1}$.

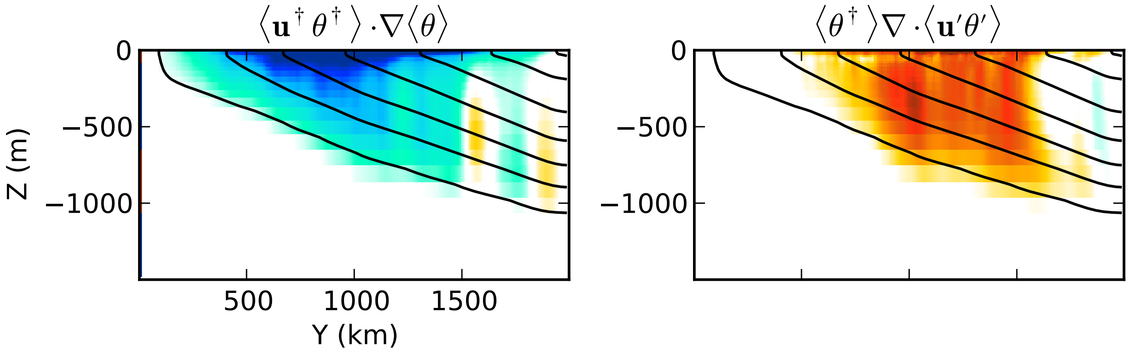

Figure 2: The dominant balance of terms in (10).

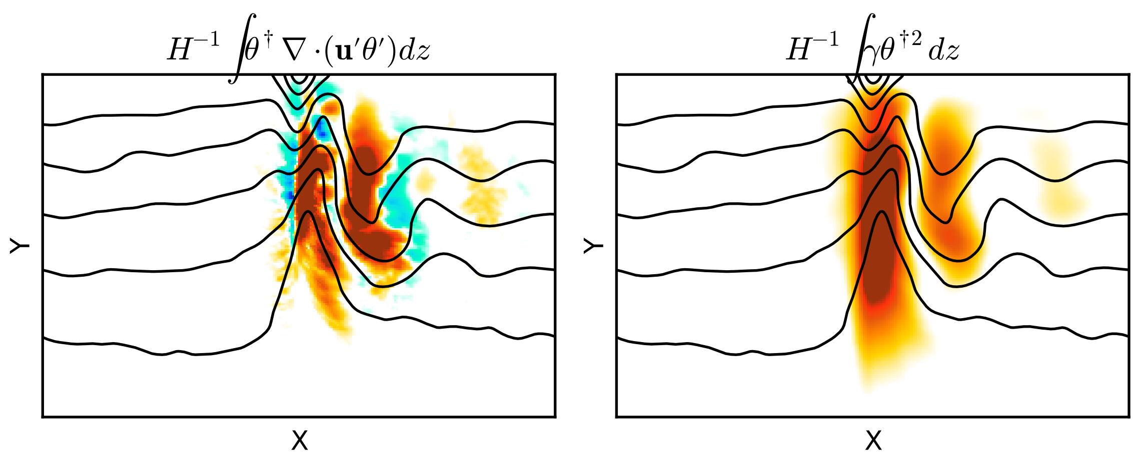

Figure 3: Comparison of the spatial structure of the transient eddy term in the (10) with a linear damping of standing wave variance.