SIO 214A Homework 3 Answers

1.) Assuming $\mathbf{u} = \nabla\phi$, the continuity equation takes the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\nabla\phi) = 0 \tag{1}$$

and the momentum equations take the form

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right) + \frac{1}{\rho}\nabla p = 0 \tag{2}$$

As in the constant density case, we want to get (2) into the form

$$\nabla\left(\mathit{something}\right) = 0 \tag{3}$$

This requires us to express

$$\frac{1}{\rho}\nabla p = \nabla\left(\mathit{something}\right) = 0 \tag{4}$$

or, in other words, we must show that dp/ρ is an exact differential. But

$$\frac{1}{\rho}\nabla p = \nabla\left(\frac{p}{\rho}\right) - p\nabla\left(\frac{1}{\rho}\right) \tag{5}$$

$$= \nabla\left(\frac{p}{\rho}\right) + \frac{p}{\rho^2}\nabla\rho \tag{6}$$

$$= \nabla\left(\frac{p}{\rho}\right) + \frac{dE}{d\rho}\nabla\rho \tag{7}$$

$$= \nabla\left(\frac{p}{\rho} + E(\rho)\right) \tag{8}$$

where we have used $p = \rho^2 dE/d\rho$. Hence (2) becomes

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{p}{\rho} + E(\rho)\right) = 0 \tag{9}$$

which implies

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{p}{\rho} + E(\rho) = 0 \tag{10}$$

after absorbing an arbitrary function of time into ϕ. Compressible, irrotational, barotropic flow is governed by (1) and (10). These equations are more complicated and much harder to solve than the corresponding equations for incompressible, irrotational flow, but they can be used to study flows with large Mach numbers.

2.) The analytic function corresponding to a source at $z = -a$ and a sink of equal strength at $z = +a$ is

$$f(z) = A[\ln(z + a) - \ln(z - a)] \tag{11}$$

where A is proportional to the source.

To determine A in terms of the given data, we reason as follows: An isolated source of the form $\phi = A \ln r$ would introduce a volume per unit time of $2\pi Ad$ where d is the depth of the lake, if the velocity field is assumed uniform with depth. A source placed at the mouth of a river would introduce a volume/time half as large. Hence $\pi Ad = Q$, where Q is the given input (gallons per second, say) of the river. Thus our anticipated solution is

$$f(z) = \phi - i\psi = \frac{Q}{\pi d} \; [\ln(z + a) - \ln(z - a)] \tag{12}$$

The river width w does not appear, because the river has been abstracted as a line.

For simplicity, we now drop the constant prefactor in (12). It is not needed to answer the rest of the question.

The real part of (12) is the velocity potential

$$\phi(x, y) = \ln|z + a| - \ln|z - a| = \frac{1}{2} \ln\left(\frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}\right) \tag{13}$$

The boundary conditions are no-normal-flow on $x^2 + y^2 = a^2$, except at the river mouths, where the appropriate conditions have already been satisfied. The no-normal-flow boundary condition requires that ψ be uniform on shoreline locations between the two river mouths, *or* that the normal derivative of ϕ vanish at the shoreline. Either one of these is sufficient, because the Cauchy-Riemann conditions then imply the other.

By the divergence theorem, $\nabla^2\phi = 0$ implies

$$\oint \frac{\partial \phi}{\partial n} = 0 \tag{14}$$

which may be regarded as a consistency condition on the boundary conditions. It requires the source at one river mouth to balance the sink at the other river mouth.

The velocity field can be computed as $(u, v) = (\phi_x, \phi_y)$ or more simply from

$$\frac{df}{dz} = u - iv = \frac{1}{z+a} - \frac{1}{z-a} = -\frac{2a}{z^2 - a^2} \tag{15}$$

To see that the boundary condition $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ is obeyed, one could verify that

$$\nabla\phi \cdot \nabla(x^2 + y^2 - a^2) = 0 \tag{16}$$

or proceed more simply as follows. The no-normal-flow boundary condition is equivalent to

$$(u, v) \cdot (x, y) = Re[(u - iv)(x + iy)] = Re\left(\frac{df}{dz}z\right) = 0 \quad \text{on} \quad z = a^{i\theta} \tag{17}$$

But on $z = a^{i\theta}$

$$\frac{df}{dz}z = -\frac{2}{a}\,\frac{e^{i\theta}}{e^{2i\theta} - 1} = -\frac{2}{a}\,\frac{1}{e^{i\theta} - e^{-i\theta}} = -\frac{2}{a}\,\frac{1}{2i\sin\theta} \tag{18}$$

which is pure imaginary. Hence the boundary condition (17) is satisfied.

To find the minimum current speed, we compute

$$u^2 + v^2 = |\frac{2a}{z^2 - a^2}|^2 \propto \frac{1}{(z^2 - a^2)(\bar{z}^2 - a^2)} \tag{19}$$

where $\bar{z} = x - iy$ is the complex conjugate of z. The minimum of (19) occurs at the *maximum* of

$$D \equiv (z^2 - a^2)(\bar{z}^2 - a^2) = r^4 + a^4 - 2a^2r^2\cos(2\theta) \tag{20}$$

where we have set $z = re^{i\theta}$. The last term in (20) is a maximum when $\theta = \pm\pi/2$. When $\theta = \pm\pi/2$,

$$D = r^4 + a^4 + 2a^2r^2 \tag{21}$$

which increases with r, and is therefore a maximum within the lake at $r = a$. Therefore the velocity is minimal at the points $z = ae^{\pm i\pi/2}$, which lie at the north and south extremity of the lake. By (19) the minimum current speed is

$$\frac{Q}{\pi d}|\frac{2a}{(ia)^2 - a^2}| = \frac{Q}{\pi da} \tag{22}$$

where we have restored the constant prefactor.