SIO 214A Homework 4

Please hand in on paper, in class, on Tuesday, November 30

1.) The equations governing N point vortices are

$$\frac{dx_i}{dt} = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{r_{ij}^2}(y_i - y_j), \quad \frac{dy_i}{dt} = \frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{r_{ij}^2}(x_i - x_j) \tag{1}$$

where Γ_i is the vorticity of the i-th vortex at (x_i, y_i), and r_{ij} is the distance between the i-th and j-th vortex. Show that (1) can be written in the canonical form

$$\Gamma_i\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \Gamma_i\frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{2}$$

where

$$H = -\sum_{i>j}\frac{\Gamma_i\Gamma_j}{2\pi}\ln r_{ij} \tag{3}$$

(Thus H is the Hamiltonian of the point vortex system. Note that each vortex pair appears only once in the sum.) Show that (2) implies

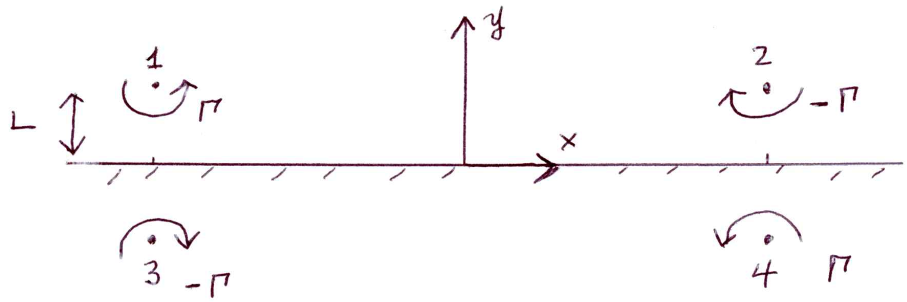
$$\frac{dH}{dt} = 0 \tag{4}$$

no matter what the specific form of H. Then show that the point vortex system also conserves the following three quantities:

$$M_x = \sum_i \Gamma_i x_i \tag{5}$$

$$M_y = \sum_i \Gamma_i y_i \tag{6}$$

$$\Omega = \sum_i \Gamma_i(x_i^2 + y_i^2) \tag{7}$$



2.) Using the invariants discovered in the previous problem, investigate the 'collision' of the two vortices, 1 and 2, with vorticities $+\Gamma$ and $-\Gamma$, moving along a straight coastline as shown. To satisfy the no-normal-flow boundary condition at the coast, we introduce the image vortices 3 and 4. Initially, when the the two vortices are still very far apart, they are both a distance L offshore.

Determine the *path* of vortices 1 and 2 as they 'collide' and move offshore as a counter-rotating pair. That is, find the curves $f_1(x_1, y_1)$ and $f_2(x_2, y_2)$ along which these two vortices move.

Remark: Wave-breaking in the surf zone produces vortices like these, and this problem could be considered as a rough illustration of how the vortices thus created produce a transient rip current (narrow current flowing offshore).