SIO 214A Midterm Selected Answers

10.) $c = \omega/k$ is the sound speed. The pressure in the large pipe is

$$p_1(x,t) = p_0 \cos[k(x-ct)] + R[k(x+ct)] \tag{1}$$

where the function $R(\cdot)$ remains to be determined. The pressure in the small pipe is

$$p_2(x,t) = T[k(x-ct)] \tag{2}$$

where $T(\cdot)$ must also be determined. To determine R and T we set $p_1(0,t) = p_2(0,t)$, i.e.

$$p_0 \cos(-\omega t) + R(+\omega t) = T(-\omega t) \tag{3}$$

and $Au_1(0,t) = \frac{A}{2}u_2(0,t)$, i.e.

$$A\left[p_0 \cos(-\omega t) - R(+\omega t)\right] = \frac{A}{2}T(-\omega t) \tag{4}$$

The minus sign before R in the last equation is associated with the fact that the pressure is in phase with the fluid velocity in the direction of wave propagation.

Since (3-4) must hold for all time, it follows that

$$T(s) = \frac{4}{3}p_0 \cos(s) \tag{5}$$

and

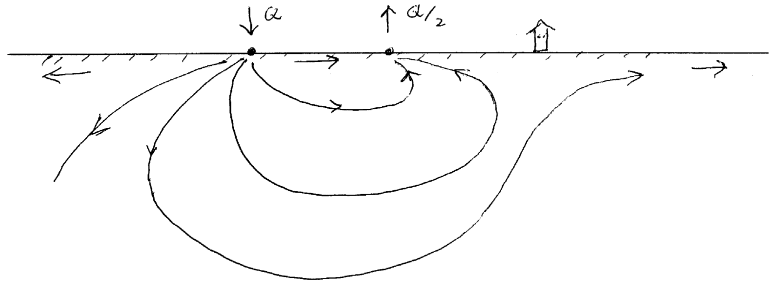
$$R(s) = \frac{1}{3}p_0 \cos(s) \tag{6}$$

for any dummy argument s. Thus

$$p_1(x,t) = p_0 \cos(kx - \omega t) + \frac{1}{3}p_0 \cos(kx + \omega t) \tag{7}$$

and

$$p_2(x,t) = \frac{4}{3}p_0 \cos(kx - \omega t) \tag{8}$$



11.) As in the lake flow problem, there is a point source of mass at the point of injection, and a point sink at the point of extraction. Now, however, we are in three dimensions, where a source strength of q (volume per second) would generate a radially symmetric outward flow given by $q = 4\pi r^2 u_r(r)$, where $u_r(r)$ is the outward radial velocity at a distance r from the source. Since the given Q represents the volume per second pumped into the *ground*—a semi-infinite domain—we have $Q = 2\pi r^2 u_r(r)$ for the polluting source. Therefore the velocity field induced by the polluting plant is

$$\mathbf{u}(x, y, z) = (u, v, w) = \frac{Q}{2\pi r^2}\hat{\mathbf{r}} = \frac{Q}{2\pi}\,\frac{1}{(x^2 + y^2 + z^2)}\frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} \tag{9}$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. Thus

$$u(x, 0, 0) = \frac{Q}{2\pi}\,\frac{1}{x^2}\frac{x}{|x|} \tag{10}$$

is the velocity field, on the line between the plants, induced by the polluting plant. By similar logic, the velocity field induced by the extracting plant is

$$u(x, 0, 0) = -\frac{1}{2}\,\frac{Q}{2\pi}\,\frac{1}{(x - L)^2}\frac{x - L}{|x - L|} \tag{11}$$

The *total* groundwater velocity field to the *right* of both plants is

$$u(x, 0, 0) = \frac{Q}{2\pi}\,\left[\frac{1}{x^2} - \frac{1}{2}\frac{1}{(x - L)^2}\right] \tag{12}$$

Setting (12) to zero, we obtain the quadratic equation

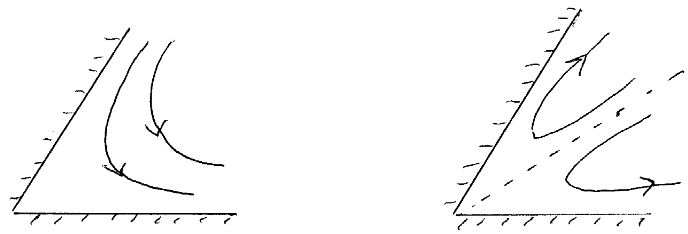
$$x^2 - 4Lx + 2L^2 = 0 \tag{13}$$

with solution

$$x = (2 + \sqrt{2})L \approx 3.14L \tag{14}$$

[The other root is at an $x < L$ to which (12) does not apply.] This is the location at which the groundwater velocity at $z = 0$ vanishes, as shown in the sketch.

The flow can be viewed as the *sum*—this is a linear problem—of a source/sink pair with strengths $\pm Q/2$, and a source of strength $+Q/2$. The source/sink pair corresponds to a dipole, and the leftover source corresponds to a monopole. At great distances only the monopole remains. The fact that the dipole decays more rapidly than the monopole is what led the homebuilder to suspect that they would cancel at just the right distance.



12.) Accepting the hint, we investigate

$$f(z) = Az^\alpha = A(re^{i\theta})^\alpha = A[r^\alpha \cos(\alpha\theta) + ir^\alpha \sin(\alpha\theta)] \tag{15}$$

The corresponding stream function is

$$\psi(r, \theta) = -Ar^\alpha \sin(\alpha\theta) \tag{16}$$

This must take a constant value along the boundaries at $\theta = 0$ and $\theta = \pi/3$. This can only be true if $\sin(\alpha\pi/3) = 0$. Therefore $\alpha/3$ must be an integer multiple of π.

The left sketch shows $\psi(r, \theta)$ for the case $\alpha = 3$ and $A > 0$. A negative value of A would reverse the arrows.

However, it is also possible to have $\alpha = 6$, etc. For $\alpha = 6$ the flow looks like the right sketch, with a 'virtual boundary' at $\theta = \pi/6$. Again, flipping the sign of A would reverse the arrows.