SIO 214A Fall 2021

Lecture Notes

The equations for a perfect barotropic fluid (Lectures 1 & 2)

We start with the abstract idea of a fluid continuum and derive the equations for an ideal, barotropic fluid, first by considering a control volume (Kundu, 5th ed., p. 96-103), and then by averaging over molecular motions (LGFD, p. 16-19), finally arriving at the equations for a perfect barotropic fluid,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} + \frac{\partial (w\rho)}{\partial z} = 0$$
$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \frac{\partial (u\, \rho \mathbf{u})}{\partial x} + \frac{\partial (v\, \rho \mathbf{u})}{\partial y} + \frac{\partial (w\, \rho \mathbf{u})}{\partial z} = -\nabla p$$
$$p = F(\rho)$$

where $\mathbf{u} = (u, v, w)$ is the velocity and $F(\rho)$ is a prescribed function. The momentum equation can also be written

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

is the time derivative following a fluid particle. These equations will serve us well for the first half of the course. From time to time we will add gravity, and eventually we will need to include the viscosity, which involves adding the term $\nu \nabla^2 \mathbf{u}$ to the momentum equation.

Here is the 'molecular derivation' of the continuity equation in greater detail: Define

$$\rho(\mathbf{x}, t) = \sum_i m_i R\left(r_i(\mathbf{x}, \mathbf{x}_i(t))\right)$$

where

$$r_i = |\mathbf{x} - \mathbf{x}_i(t)|$$

and $R(r)$ is a 'sampling function', maximum at $r = 0$, and vanishing when $r > r_0$. It must be normalized such that

$$4\pi \int r^2 R(r) dr = 1$$

to ensure that

$$\iiint \rho(\mathbf{x},t)d\mathbf{x} = \sum_i m_i$$

Then computing

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(\mathbf{x},t) &= \sum_i m_i R'(r_i)\frac{\partial}{\partial t}r_i(\mathbf{x},\mathbf{x}_i(t)) \\
&= \sum_i m_i R'(r_i)\left(\frac{\partial r_i}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial r_i}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial r_i}{\partial z_i}\frac{dz_i}{dt}\right) \\
&= \sum_i m_i R'(r_i)\left(-\frac{\partial r_i}{\partial x}\frac{dx_i}{dt} + -\frac{\partial r_i}{\partial y}\frac{dy_i}{dt} + -\frac{\partial r_i}{\partial z}\frac{dz_i}{dt}\right) \\
&= \sum_i m_i R'(r_i)\left(-\nabla r_i \cdot \frac{d\mathbf{x}_i}{dt}\right) \\
&= -\sum_i m_i \left(\nabla R(r_i) \cdot \frac{d\mathbf{x}_i}{dt}\right) \\
&= -\nabla \cdot \sum_i m_i \frac{d\mathbf{x}_i}{dt} R(r_i) \\
&\equiv -\nabla \cdot (\rho\mathbf{u}(\mathbf{x},t))
\end{aligned}$$

if we take the last line as the definition of velocity.

Static stability (Lecture 3)

The simplest solutions are states of rest ($\mathbf{u} = 0, \partial/\partial t = 0$). To make these interesting, we must add gravity. In the state of rest, $p = p(z)$, $\rho = \rho(z)$ and the barotropic equations reduce to

$$0 = -\frac{dp}{dz} - \rho g \tag{1}$$

$$p = F(\rho) \tag{2}$$

For a given total mass of fluid, the solution is unique.

In this lecture we diverge from the main line of the course by considering the more realistic case in which F depends on two state variables. A natural choice for the second state variable is temperature, but entropy proves more convenient. Thus

$$p = F(\rho, \eta) \tag{3}$$

where η is the entropy per unit mass. The entropy proves convenient because the needed extra equation is a simple one:

$$\frac{D\eta}{Dt} = 0 \tag{4}$$

Entropy is conserved in adiabatic flow. The addition of a second state variable means that there are an infinite number of possible hydrostatic flows. However, only some of them are stable with respect to small disturbance.

First we prove *Archimedes' principle*: The pressure force on an immersed body is equal to the weight of the fluid it displaces. This result applies to any type of fluid, because it only uses the hydrostatic relation, eqn (1). For a fluid parcel displaced vertically, Archimedes' principle leads to the prediction that the state of rest is stable if

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} + \frac{g}{c^2} < 0 \quad \text{(stable)} \tag{5}$$

where

$$c^2 = \frac{\partial}{\partial \rho} F(\rho, \eta) \tag{6}$$

and c turns out to be the sound speed (next lecture). From this we see that the effect of fluid compressibility is always a de-stabilizing one. Eqn (5) a general result, but meteorologists prefer equivalent statements that refer to temperature and make use of the ideal gas relations. One such form is

$$\frac{\partial T}{\partial z} + \frac{g}{C_p} > 0 \quad \text{(stable)} \tag{7}$$

and another is

$$\frac{\partial \theta}{\partial z} > 0 \quad \text{(stable)} \tag{8}$$

where θ is the *potential temperature*. Potential temperature is a poor indicator of stability in the ocean, because seawater density depends on temperature *and* salinity. For this lecture, see LGFD p. 39-42. For a good, and very gentle, introduction to atmospheric thermodynamics and moisture variables, see *Introduction to Theoretical Meteorology* by Seymour Hess, 1959.

Sound waves and the incompressible limit (Lecture 4, 5, 6, and 7)

From now on, to keep things as simple as possible, we omit the gravity term.

We want to focus on flows in which the mass density is constant. However, it is not a simple matter of setting $\rho = \rho_0$ in the full equations. If we do, the equation $p = F(\rho)$ implies that $p = F(\rho_0)$ (constant), and the remaining equations become

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{D\mathbf{u}}{Dt} = 0 \tag{2}$$

Eqns (1-2) are ill posed; there are more equations than unknowns.

The subtlety here is that $\rho \to \rho_0$ corresponds to the limit $c \to \infty$ where c is the sound speed. A careful application of this limit will yield the equations

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla p \tag{4}$$

$$p = F(\rho) \tag{5}$$

Eqns (3-4) form closed set of equations for $\mathbf{u}$ and p in which ρ does not appear. In these equations, the pressure acts to enforce the constraint that the velocity field be nondivergent. Eqn (5) is then best viewed as an 'auxiliary equation' that gives you ρ after the fact, that is, after (3-4) have been solved for $\mathbf{u}$ and p. However, as $c \to \infty$, the difference between ρ and ρ_0 becomes so small that it is hardly worth knowing.

If you are a meteorologist, you might be offended at the idea that constant density is an acceptable approximation. However, meterologists end up with a set of equations that are not so very different from (3-4) by the strategy of adopting pressure coordinates, that is, by replacing z with p as the vertical coordinate. To get the flavor of this, see LGFD p. 102-105. For a more thorough introduction, see the beautiful article entitled "A view of the equations of meteorological dynamics and various approximations" by A. A. White in *Large-Scale Atmosphere-Ocean Dynamics*, edited by Norbury & Roulstone, Cambridge, 2002.

We proceed by considering a solution of the full equations for which we can take the limit $c \to \infty$. Unfortunately, since we are dealing with coupled nonlinear PDEs, we cannot solve the full equations exactly. We therefore assume that the motion is a slight departure from the state of rest,

$$\rho = \rho_0, \quad \mathbf{u} = 0 \tag{6}$$

That is, we assume that

$$\rho = \rho_0 + \rho'(\mathbf{x}, t) \quad \mathbf{u} = 0 + \mathbf{u}'(\mathbf{x}, t) \tag{7}$$

where ρ' and $\mathbf{u}'$ are small. Then, neglecting the products of small quantities, we obtain the linearized form of the full equations, namely

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0 \tag{8}$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p \tag{9}$$

$$p = F(\rho_0) + F'(\rho_0)\rho' \equiv p_0 + c^2 \rho' \equiv p_0 + p' \tag{10}$$

Combining equations we obtain the 'wave equation'

$$\frac{\partial^2 \rho'}{\partial t^2} = c^2 \nabla^2 \rho' \tag{11}$$

This is a standard equation of mathematical physics which you have likely seen before (see LPDE Chapter 6 or the book by G. B. Whitham, *Linear and Nonlinear Waves*, Chapter 7).

In one space dimension, u' and p' obey the same equation as ρ'. Thus

$$\frac{\partial^2 u'}{\partial t^2} = c^2 \frac{\partial^2 u'}{\partial x^2} \tag{12}$$

This has the general solution (d'Alembert's solution)

$$u' = F(t - x/c) + G(t + x/c) \tag{13}$$

where $F(\cdot)$ and $G(\cdot)$ are arbitrary functions of a single variable.

Consider an infinite pipe along the x-axis, with fluid filling the right half of the pipe, and a movable piston located at $x_p(t)$ which is under our control. To determine the response of the fluid to the motion of the piston we must solve (12) subject to the boundary condition

$$u'(x_p(t), t) = \frac{d}{dt} x_p(t) \equiv u_p(t) \tag{14}$$

If the piston never moves far from its initial location at $x = 0$, we may approximate this boundary condition as

$$u'(0, t) = u_p(t) \tag{15}$$

If there are no other sources of excitation besides the piston, then there can be no incoming wave, and it follows that $G \equiv 0$. (This type of boundary condition is called a *radiation condition*.) Then (15) implies that $F(t) = u_p(t)$ and hence

$$u'(x,t) = u_p(t - x/c) \tag{16}$$

The fluid velocity at (x,t) is just the piston velocity at the earlier time $t - x/c$. The signal moves at the sound speed c. From (8) and (10) we determine

$$p' = \rho_0 c u_p(t - x/c) \tag{17}$$

and

$$\rho' = \frac{\rho_0}{c} u_p(t - x/c) \tag{18}$$

Thus the density perturbation is smaller than the mean density by a factor of the Mach number. As $c \to \infty$, for all finite x,

$$u' \to u_p(t) \tag{19}$$

$$p' \to \rho_0 c u_p(t) \tag{20}$$

$$\rho' \to \frac{\rho_0}{c} u_p(t) \tag{21}$$

and the fluid in the pipe moves as a solid slug, obeying the one-dimensional condition for zero divergence,

$$\frac{\partial u}{\partial x} = 0 \tag{22}$$

This is classic action-at-a-distance, but it is true only as an approximation. In reality, nothing goes faster than the speed of sound. The sound speed plays the same role role in fluid dynamics as does the speed of light in electrodynamics. A good book for sound waves is *Waves in Fluids* by James Lighthill.

The one-dimensional case is interesting, but to get a good idea of what is going on, we need to consider more space dimensions. For simplicity, we consider the two-dimensional case. The basic question is this: In what sense are (3-4) a good approximation to the full equations,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \tag{23}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p \tag{24}$$

$$p = F(\rho) \tag{25}$$

Again we regard the primes as the departure from the state of rest, but now we do not assume that they are small. The full equations (including sound waves) take the form

$$\frac{\partial\rho'}{\partial t} + \rho_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = -\nabla\cdot(\rho'\mathbf{u}') \approx 0 \qquad (26)$$

$$\rho_0\frac{\partial u'}{\partial t} + c^2\frac{\partial\rho'}{\partial x} = -\frac{\partial(\rho' u')}{\partial t} - \nabla\cdot[(\rho_0+\rho')u'\mathbf{u}'] \approx -\nabla\cdot[\rho_0 u'\mathbf{u}'] \equiv \mathcal{N}_x \qquad (27)$$

$$\rho_0\frac{\partial v'}{\partial t} + c^2\frac{\partial\rho'}{\partial y} = -\frac{\partial(\rho' v')}{\partial t} - \nabla\cdot[(\rho_0+\rho')v'\mathbf{u}'] \approx -\nabla\cdot[\rho_0 v'\mathbf{u}'] \equiv \mathcal{N}_y \qquad (28)$$

where we have placed all the linear terms on the left-hand side. In the nonlinear terms on the right-hand side, we have introduced a further approximation that anticipates that ρ' will be small.

Next we introduce a change of variables that will allow us to better see what is going on. We let

$$u' = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y} \qquad (29)$$

$$v' = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} \qquad (30)$$

Instead of (u', v', ρ'), it will be better to use (ρ', ϕ, ψ) as independent variables. We call ϕ the velocity potential and ψ the stream function. The significance of these new variables is that the divergence

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = \nabla^2\phi \qquad (31)$$

is the Laplacian of ϕ; and the vorticity

$$\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \nabla^2\psi \qquad (32)$$

is the Laplacian of the stream function.

[Digression. This is the first appearance of vorticity in our course. To give a physical interpretation of vorticity, I will digress to discuss the decomposition of the velocity field into components of divergence, vorticity, and strain. Kundu, p. 77-82, does this for the general 3d case; my presentation will be simpler, being confined to 2 dimensions.]

Taking the divergence and curl of (27-28), we arrive at the equations

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla^2 \phi = 0 \tag{33}$$

$$\rho_0 \frac{\partial \nabla^2 \phi}{\partial t} + c^2 \nabla^2 \rho' = \frac{\partial \mathcal{N}_x}{\partial x} + \frac{\partial \mathcal{N}_y}{\partial y} \tag{34}$$

$$\frac{\partial \nabla^2 \psi}{\partial t} = \frac{\partial \mathcal{N}_y}{\partial x} - \frac{\partial \mathcal{N}_x}{\partial y} \tag{35}$$

In the linear limit these become

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla^2 \phi = 0 \tag{36}$$

$$\rho_0 \frac{\partial \nabla^2 \phi}{\partial t} + c^2 \nabla^2 \rho' = 0 \tag{37}$$

$$\frac{\partial \nabla^2 \psi}{\partial t} = 0 \tag{38}$$

which are in the form of equations for normal modes. The variables ρ' and ϕ representing the 'sound-wave mode' with dispersion relation $\omega = ck$. ψ represents the 'vorticity mode' with dispersion relation $\omega = 0$. We think of (ρ', ϕ) as the 'fast mode' and ψ as the slow mode. In the linear limit these two modes are fully decoupled. The sound waves and the slow vortical motion are completely unaware of one other.

In the full dynamics (33-35), the nonlinear terms couple the two modes together. Symbolically,

$$\mathcal{N} = \phi * \phi + \phi * \psi + \psi * \psi \tag{39}$$

where $\mathcal{N}$ stands for $\mathcal{N}_x$ or $\mathcal{N}_y$. However this coupling tends to be weak because the timescale of the fast mode (ρ', ϕ) is so much shorter than that of the slow mode ψ.

If you are only interested in the slow mode, then a reasonable approximation is to throw away the equations (33) and (34) for the fast mode, and to keep only the terms $\psi * \psi$ in the slow-mode equation (35). The result is (35) in the form

$$\frac{\partial \nabla^2 \psi}{\partial t} = -(\mathbf{u}_\psi \cdot \nabla)(\nabla^2 \psi) \tag{40}$$

where $\mathbf{u}_\psi = (-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial y})$ is the slow velocity. Equation (40) describes the slow mode interacting with itself, and is equivalent to the 'sound-free' dynamics

(3-5). In other words, the replacement of the full, compressible equations by (3- 5) is equivalent to just throwing out the fast modes. In two dimensions, it is generally easier to use (40) than to use (3-4). In three dimensions it may be easier to use (3-4) than to use the (more complicated) three-dimensional analog of (40). However, the two approaches—vorticity equation versus momentum equations—are equivalent, and always give the same result.

The elimination of the fast modes converts the dynamics from an 'exact' dynamics in which nothing moves faster than the speed of sound, to an approximate dynamics with action-at-a-distance: In the formulation (40), one must determine ψ from $\nabla^2\psi$; in the formulation (3-4), one must determine p from

$$\nabla^2 p = -\nabla \cdot (\rho_0(\mathbf{u} \cdot \nabla)\mathbf{u}) \tag{41}$$

Elliptic equations always express action-at-a-distance.

By solving for the slow motion by itself, we are not necessarily assuming that the sound waves are negligible. We are actually only saying that we don't care about them. It doesn't matter what they are doing, because they couple only weakly to what we *do* care about.

But suppose we start from a situation in which no fast modes are present ($\rho' \equiv 0 \equiv \phi$). Would the fast modes remain zero? No. Nonlinear terms of the form $\psi * \psi$ on the right hand side of (34) would excite fast modes. How effectively would they do that? This was considered in a famous paper by James Lighthill. He found that the excitation of sound waves by the slow vortical modes was extremely weak. The weakness is due to the tremendous mismatch in the time scales. The ratio of time scales is the Mach number, U/c.

If we were to re-admit gravity—we won't—this entire discussion repeats itself at another level. Internal gravity waves become present, and they become the new fast modes (but not as fast as sound waves). Throwing out the gravity modes leaves us with a new, slow, vortical mode interacting only with itself. This new type of slow dynamics—with both sound waves and gravity waves extracted—is called *quasigeostrophic dynamics*. But here is the catch. The phase speed of internal waves is not that much greater than the velocity of the quasigeostrophic mode. The ratio of time scales is the Froude number U/c_{grav}, where now c_{grav} is the speed of gravity waves. Thus the coupling between gravity modes and vortical modes is not a weak one. These modes interact strongly, and their interaction is a hot research topic in both meteorology and oceanography.

Potential flow (Lecture 9 & 10)

Soon after the perfect-fluid equations were discovered, several people had the idea to seek solutions in the form

$$\mathbf{u} = \nabla\phi \tag{1}$$

Flows with the property (1) are called *potential flows*. Most fluids books treat them thoroughly (e.g. Kundu, chapter 6) so these notes offer only an outline.

The primary motivation for (1) was that it made things much easier. Early workers justified (1) by noting that fluids that were set in motion *solely* by pressure forces satisfied (1) automatically. Nevertheless, results obtained with (1) often disagreed with reality. The need to go beyond (1) became clear following Helmholtz's great paper of 1885.

The assumption (1) is deficient because it assumes that the vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$ vanishes. Most flows contain vorticity. However, viscosity is required to generate the vorticity, and, however generated, vorticity behaves unrealistically unless viscosity is present to control it. Thus, to go beyond (1), we must introduce both viscosity and vorticity. However, the presence of vorticity and viscosity in the equations greatly complicates their analysis. We follow the historical development of the field by learning as much as we can from the consequences of (1) before adding vorticity and viscosity.

We start with the constant-density equations discussed in previous lectures. With the assumption (1) the continuity equation $\nabla \cdot \mathbf{u} = 0$ takes the form

$$\nabla^2\phi = 0 \tag{2}$$

and the momentum equations take the form of the Bernoulli equation,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{p}{\rho_0} = 0 \tag{3}$$

Eqns (2-3) are two coupled equations in the two unknowns $\phi(x, y, z, t)$ and $p(x, y, z, t)$. They replace the *four* coupled equations in (u, v, w, p). The only nonlinearity resides in the middle term of (3).

How easy is it to solve (2-3)? That depends on the boundary conditions, which largely control the solutions. If the boundary conditions on the Laplace eqn (2) involve only ϕ, then that equation determines $\phi(x, y, z, t)$ and (3) serves only to tell us $p(x, y, z, t)$. The nonlinearity in (3) is no problem, because we know $\phi(x, y, z, t)$ from having solved (2).

If the boundary conditions involve *both* ϕ and p—as in the surface-wave problem—then things become much more difficult. People have been working hard on surface waves for more than two centuries.

Although (2) and (3) apply to general three-dimensional flow, it is much easier to discuss solutions in two space dimensions, where analytic function theory becomes a powerful tool. In two dimensions,

$$\nabla \cdot \mathbf{u} = 0 \Rightarrow \mathbf{u} = (-\psi_y, \psi_x) \tag{4}$$

and

$$\nabla \times \mathbf{u} = 0 \Rightarrow \mathbf{u} = (\phi_x, \phi_y) \tag{5}$$

Thus we may describe the velocity field using *either* the stream function or the velocity potential. (Contrast this with an earlier lecture in which we needed *both*.) The two are related by

$$\phi_x = -\psi_y, \quad \phi_y = \psi_x \tag{6}$$

from which we see that curves of constant ϕ are everywhere perpendicular to curves of constant ψ. Moreover, since vanishing vorticity implies

$$\nabla^2 \psi = 0 \tag{7}$$

both ϕ and ψ are harmonic functions.

We still need a method for finding ϕ or ψ. Analytic function theory provides such a method. Let $z = x + iy$ and consider functions $f(z)$ for which the derivative of $f(z)$ has a sensible, 'direction independent', meaning. From this requirement it follows that the real and imaginary parts of

$$f(z) = \phi(x, y) - i\psi(x, y) \tag{8}$$

obey (2), (6) and (7), and therefore represent a possible potential flow. In other words, every analytic function corresponds to a solution of the potential-flow problem in two dimensions. The challenge then becomes: Can you find an analytic function that satisfies your boundary conditions? Typical boundary conditions are no-normal-flow at a prescribed solid boundary. Then the challenge becomes: Can you find an analytic function whose imaginary part—the stream function—takes a constant value along your particular boundary.

The proper choice of analytic function is an art best learned by practice. Two facts are of immense importance: First, the sum of analytic functions is an analytic function. Second, the analytic function of an analytic function is an analytic function. The latter corresponds to conformal mapping.

The problem with all this is that potential flows are fundamentally lacking in personality. If you are having a party, and you only invite potential flows, you are in for a dull evening. To liven things up, you need a small amount of vorticity. But be careful with vorticity. If you invite too much vorticity, you can expect broken furniture (aka turbulence) and a visit from the police.

Vorticity without viscosity (Lecture 11)

References for this are LGFD pp. 197-205, or chapter 5 of Kundu. This topic is thoroughly covered in all fluids books, so the following is merely a summary of the lecture.

The vorticity is defined as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{1}$$

and carries the physical interpretation of local solid rotation. To derive its evolution equation from the incompressible Euler equations

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p \tag{3}$$

we use a vector identity to rewrite the latter as

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} = -\nabla \left(p + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) \tag{4}$$

Then, taking the curl and invoking another vector identity, we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} \tag{5}$$

In two space dimensions $\mathbf{u} = (u, v, 0)$ and $\boldsymbol{\omega} = (0, 0, \zeta)$, where $\zeta = v_x - u_y$. In this case (5) reduces to

$$\frac{\partial \zeta}{\partial t} + (\mathbf{u} \cdot \nabla)\zeta = 0 \tag{6}$$

Thus vorticity is conserved on fluid particles in two-dimensional flow. The general three-dimensional equation (5) is much harder to interpret.

Helmholtz (1858) was first to see what it meant. He considered the relative displacement $\delta\mathbf{r}(t)$between two infinitesimally separated fluid particles located at $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Using Taylor series expansion,

$$\frac{d}{dt}\delta\mathbf{r}(t) = \mathbf{u}(\mathbf{r}_2(t),t) - \mathbf{u}(\mathbf{r}_1(t),t) \tag{7}$$

implies

$$\frac{d}{dt}\delta r_i = \frac{\partial u_i}{\partial x_j}\delta r_j \tag{8}$$

in index notation. Compare this to (5) rewritten as

$$\frac{D\omega_i}{Dt} = \frac{\partial u_i}{\partial x_j}\delta\omega_j \tag{9}$$

We see that the vorticity vector undergoes the same evolution as the displacement vector between two fluid particles located 'along' the vorticity vector. As the two fluid particles change their orientation, the vorticity vector changes its direction. As the fluid particles move apart, the vorticity vector gets longer in proportion.

Kelvin provided the next step. He showed that the circulation

$$C(t) \equiv \oint \mathbf{u}\cdot d\mathbf{r} \tag{10}$$

where the integral is around a closed material loop of fluid particles, is conserved:

$$\frac{dC}{dt} = 0 \tag{11}$$

Stokes's theorem,

$$\oint \mathbf{u}\cdot d\mathbf{r} = \iint \boldsymbol{\omega}\cdot\hat{\mathbf{n}}\, dA \equiv \Gamma \tag{12}$$

provides the connection with vorticity.

Consider a vortex tube, defined as the extension, at a fixed time, of a loop of particles in the direction of the local vorticity vector. Since $\nabla\cdot\boldsymbol{\omega} = 0$, Γ is uniform along the tube. By the circulation theorem, it is also constant in time. Thus, as the tube moves with the fluid particles it contains, the product of its cross section and its vorticity remains the same. Where the tube is stretched, becoming thinner, its vorticity must increase.

Vorticity Dynamics (Lecture 12)

The vorticity equation (5) can be considered an alternative formulation of fluid dynamics. That is, instead of solving (2-3), we can use (5) to step the vorticity $\boldsymbol{\omega}$ forward in time and the 'invert' (1) to find the velocity $\mathbf{u}$ at the new time.

In two dimensions this is easy, and it is probably the best way to solve two-dimensional, incompressible Euler dynamics. One uses (6) to step ζ forward in time and then 'inverts'

$$\nabla^2 \psi = \zeta \tag{13}$$

to determine ψ and hence $\mathbf{u} = (-\psi_y, \psi_x)$ at the new time.

In three dimensions, the determination of $\mathbf{u}$ from $\boldsymbol{\omega}$ is analogus to finding the magnetic field $\mathbf{B}$ from the current density $\mathbf{j}$ via

$$\nabla \times \mathbf{B} = \mathbf{j} \tag{14}$$

and the Biot-Savart law. This approach is well covered in chapter 1 of P. G. Saffman, *Vortex Dynamics*, Cambridge, 1992. You may remember it from electrodynamics. However, it is probably fair to say that in three dimensions most people prefer to solve (2-3) rather than (1) and (5).

The term *vorticity dynamics* is reserved for flows in which vorticity is present within only a very small fraction of the flow. Such flows occupy the very outward limit of analytical theory. For a good dose, see Saffman's book.

One can think of vorticity as an explosive substance. If present in small and widely separated locations, it can be handled safely, albeit with considerable exertion (Saffman's book). However, if vorticity is present throughout the flow, then chaos (aka turbulence) ensues, and the only recourse is to adopt a *statistical* approach.

The poster child of vorticity dynamics is point vortex dynamics in two dimensions (Helmholtz 1858, Kirchoff 1876). This is barely touched on in Kundu, pp. 187-191, but it turns out to be a very useful way of visualizing two-dimensional flow. For a thorough introduction, see H. Aref, *Point vortex dynamics: a classical mathematics playground* J. Mathematical Physics, vol 48, 2007.

Consider a point vortex at the origin. Its total, integrated vorticity Γ occupies an infinitesimally small region near $x = y = 0$. Outside this region the vorticity vanishes. Thus

$$\nabla^2 \psi = \Gamma \, \delta(\mathbf{x}) \tag{15}$$

where $\delta(\mathbf{x})$ is the delta function. Using the divergence theorem we find that

$$\psi(\mathbf{x}) = \frac{\Gamma}{2\pi}\ln|\mathbf{x}| = \frac{\Gamma}{4\pi}\ln\left(x^2+y^2\right) \tag{16}$$

The associated velocity field is

$$\mathbf{u}(\mathbf{x}) = (-\psi_y, \psi_x) = \frac{\Gamma}{2\pi r^2}(-y, x) \tag{17}$$

where $r^2 = x^2 + y^2$.

In *point vortex dynamics* there are N point vortices, and they push each other around: Each vortex is advected by the velocity field induced by the other $N-1$ vortices. (A vortex has no effect upon itself.) As $N \to \infty$ the collection of point vortices behaves like a continuous distribution of vorticity governed by (6) and (13). In fact, numerical models of two-dimensional flow are sometimes formulated in terms of point vortices. If $N \leq 3$, the motion is exactly solvable (integrable), but if $N > 3$ the motion is generally chaotic, demonstrating how little vorticity is required to produce chaos.

We consider the special case $N = 2$ of two point vortices. For this we have

$$\psi(\mathbf{x}) = \frac{\Gamma_1}{2\pi}\ln|\mathbf{x}-\mathbf{x}_1| + \frac{\Gamma_2}{2\pi}\ln|\mathbf{x}-\mathbf{x}_2| \tag{18}$$

and

$$\mathbf{u}(\mathbf{x}) = \frac{\Gamma_1}{2\pi r_1^2}(y_1-y, x-x_1) + \frac{\Gamma_2}{2\pi r_2^2}(y_2-y, x-x_2) \tag{19}$$

where $r_1 = |\mathbf{x}-\mathbf{x}_1|$ and $r_2 = |\mathbf{x}-\mathbf{x}_2|$. The dynamics is

$$\frac{d\mathbf{x}_1}{dt} = \frac{\Gamma_2}{2\pi\, r_{12}^2}(y_2-y_1, x_1-x_2) \tag{20}$$

$$\frac{d\mathbf{x}_2}{dt} = \frac{\Gamma_1}{2\pi\, r_{12}^2}(y_1-y_2, x_2-x_1) \tag{21}$$

where $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$. The analysis of (20-21) is aided by the fact that the following 4 quantities remain constant in time:

$$r_{12}, \quad \Gamma_1 x_1 + \Gamma_2 x_2, \quad \Gamma_1 y_1 + \Gamma_2 y_2, \quad \Gamma_1(x_1^2+y_1^2) + \Gamma_2(x_2^2+y_2^2) \tag{22}$$

These correspond to the conservation of energy, x-direction momentum, y-direction momentum, and angular momentum, respectively. Two simple cases are co-rotating ($\Gamma_1 = \Gamma_2$) and counter-rotating ($\Gamma_1 = -\Gamma_2$) vortices.

Two-dimensional turbulence (Lecture 13)

In three space dimensions, the stretching of vortex tubes transfers energy from the large scales of motion, which are doing the stretching, to the small scales of motion associated with fluid spinning around the tube. As we shall see in a future lecture, this mechanism of energy transfer is very efficient.

In two space dimensions, it is a very different story. The energy-containing scales transfer their energy to *larger* scales. This astonishing fact was not realized until the mid-twentieth century.

The fundamental papers are by:

Onsager, L. 1949. Statistical hydrodynamics. Nuovo Cimento, Suppl. **6**, p. 279-287.

Kraichnan, R. H. 1967. Inertial ranges in two-dimensional turbulence. *Phys. Fluids* **10**, 1417-1423.

Batchelor, G. K. 1969 Computation of the energy spectrum in homogeneous two-dimensional turbulence. *Phys. Fluids* **12** , II-233-II-239.

Leith, C. E. 1968. Diffusion approximation for two-dimensional turbulence. *Phys. Fluids* **11** , 671-673.

The history behind Onsager's paper is discussed in:

Eyink, G.L. & K.R. Sreenivasan 2006. Onsager and the theory of hydrodynamic turbulence. Rev. Mod. Phys., **78**, 87-135.

The lecture follows *LGFD* pp. 217-221.

Viscosity (finally). (Lecture 14-15)

Much of this course has been concerned with the incompressible perfect-fluid equations. Now we add viscosity to the momentum equation to obtain the Navier Stokes equations,

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \tag{2}$$

with the viscous coefficient ν assumed to be constant. The momentum equation (2) can also be written

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial T_{ij}}{\partial x_j} \tag{3}$$

where

$$T_{ij} = p\delta_{ij} - \tau_{ij} \tag{4}$$

is the *momentum flux* tensor, and

$$\tau_{ij} = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{5}$$

is the *deviatoric stress tensor*. (Many books refer to $-T_{ij}$ as the *stress tensor*, so close attention to definitions is required. I will be following *LGFD* pp. 19-25 for this lecture.)

Where does (5) come from and what does it mean? First we explain what it means. If we integrate the i-th component of (3) over a fixed control volume, we obtain

$$\frac{d}{dt} \iiint dxdydz\; u_i + \cdots = -\iint dA\; T_{ij}\hat{n}_j = \iint dA\, (-p\hat{n}_i + \tau_{ij}\hat{n}_j) \tag{6}$$

where $\hat{n}_i$ is the i-th component of the outward pointing unit normal to the surface of the control volume. The ... stand for the advective fluxes of momentum that arise form the second term in (3). From (6) we see that T_{ij} represents the flux of i-direction momentum in the j-direction, and that τ_{ij} represents the part of the flux not arising from pressure. Thus τ_{ij} includes *tangential* forces acting on the surface of the control volume. We had previously neglected these tangential forces by assuming that there was only a normal force (pressure) and that it did not depend on the orientation of the surface.

We must still find a way to justify (5). There are two approaches to this. The first one, which I call the 'top down' approach, is the one followed in most fluids books. It *assumes* that the 9 components of τ_{ij} are proportional to the 9 components of $\partial u_i/\partial x_j$, and that the law relating them is invariant to coordinate system rotations. It also assumes that you know the general form of a fourth order isotropic tensor (Kundu, section 4.5), and it offers no means of estimating ν. Mathematicians love it because it is clean and easy.

The other approach, which I call 'bottom up', starts with molecules. It derives the Boltzmann equation governing the probability distribution of molecular velocity in an ideal gas, and applies the Chapman-Enskog expansion to eventually arrive at (5). A proper explanation of this method would fill a whole course. If you are interested in digging deeper into this, see the lecture notes by Paul Dellar, University of Oxford, at:

https://people.maths.ox.ac.uk/dellar/papers/MMPkinetic.pdf

The two general derivations of viscosity reflect a bitter 'culture war' that raged throughout the latter half of the nineteenth century. On the one side were the positivists, led by Ernst Mach, who, despite Dalton's Laws, argued that it was impermissible to invoke anything (such as atoms) which could never be directly observed. On the other side were the atomists, led by Ludwig Boltzmann. As the nineteenth century drew to a close, the positivists actually seemed to be winning. But they were convincingly defeated by Einstein in his 1905 paper on Brownian motion. Sadly, Boltzmann, who died in 1906, probably never realized the extent of his victory.

We shall follow a very cheap version of the 'bottom up' approach , which is sufficient to get the flavor. First we attempt to derive the momentum equation by the same molecular-averaging method that we used to derive the continuity equation in the first lecture of this course (*LGFD* pp. 19-23). We find that

$$-\tau_{xx} = \langle u'_{mol} u'_{mol} \rangle, \quad -\tau_{xy} = \langle u'_{mol} v'_{mol} \rangle, \quad -\tau_{xz} = \langle u'_{mol} w'_{mol} \rangle \tag{7}$$

where

$$\mathbf{u}'_{mol} = (u'_{mol}, v'_{mol}, w'_{mol}) \tag{8}$$

is the *departure* of the molecular velocity from the local average of molecular velocities, namely the continuum velocity $\mathbf{u}$.

Next we consider the special situation in which the average velocity takes the form

$$\mathbf{u} = (u(y), 0, 0) \tag{9}$$

For this situation (5) predicts that

$$\tau_{xy} = \nu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \nu \left(\frac{\partial u}{\partial y} \right) \tag{10}$$

Thus the flux of x-direction momentum, across the surface $y = 0$, in the direction of increasing y, is

$$-\tau_{xy} = -\iint dxdz\; \nu \left(\frac{\partial u}{\partial y} \right) \tag{11}$$

and to 'prove' (5) we must show that

$$-\nu \left(\frac{\partial u}{\partial y} \right) = \langle u'_{mol} v'_{mol} \rangle \tag{12}$$

for some scalar ν. This we do following the steps presented in *LGFD* pp. 23-26. We find that

$$\nu \approx \frac{s\lambda}{2} \tag{13}$$

where s is the speed of molecules and λ is the mean free path.

Eddy viscosity. (Lecture 16)

The course ends with a review of molecular viscosity and a parallel discussion of macroscopic averaging (often called Reynolds averaging), its associated closure problem, and the artifice of *eddy viscosity.* From the modern physics viewpoint, eddy viscosity represents a kind of *renormalized* molecular viscosity that results when one attempts to sweep the smaller scales of the continuum velocity into the same category as molecular fluctuations. Its conventional justification in terms of 'mixing length theory' is an outright embarrassment. See *LGFD* pp 31-34.

Numerical models that do not resolve the smallest scales of the velocity field *must* contain some form of eddy viscosity. Such models are now called *Large Eddy Simulations* (LES) to distinguish them from *Direct Numerical Simulations* (DNS) which resolve the viscous cutoff.

The journey ends with a discussion of the so-called *Zeroth Law of Turbulence*: that the molecular dissipation of energy is independent of ν as $\nu \to 0$, and its implication that, as $\nu \to 0$, solutions of the Navier-Stokes equations form singularities in a finite time (*LGFD* pp 226-230). A Clay Prize of \$ 1 million dollars is yours if you can prove or disprove this conjecture. See

https://www.claymath.org/millennium-problems/navierstokes-equation

and good luck!