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Hamiltonian derivation of the nonhydrostatic pressure-coordinate model

By RICK SALMON[1]* and LESLIE M. SMITH[2]
[1]*Scripps Institution of Oceanography, USA*
[2]*Yale University, USA*

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Miller (1974) and Miller and Pearce (1974) (see also Miller and White 1984) introduced a new kind of approximation in which the fluid motion—though nonhydrostatic—is governed by equations written in pressure coordinates. The Miller–Pearce (MP) equations admit buoyant convection while filtering out acoustic waves, and exactly conserve a form of energy and potential vorticity on fluid particles (Johnson 1978). However, like the more familiar anelastic equations (Ogura and Phillips 1962; Gough 1969), the MP equations incorporate a prescribed reference temperature, $T_s(p)$, and become inaccurate if the actual temperature wanders far from $T_s(p)$. As noted by White (1989), this feature of the MP equations may make them inapplicable to flows in which the temperature varies greatly on isobaric surfaces. To give another example, the MP equations could not be used to model convection in a stellar atmosphere, in which the average temperature profile is, typically, not foreknown.

In an important paper, White (1989) extended the MP model, removing the prescribed reference temperature, while retaining the conservation laws and other desirable properties of MP. White speculated that his extended equations have a Hamiltonian structure, and that "a slick derivation of the Ertel property could be constructed if the relevant Hamiltonian were known . . ." In this note, we derive White's extended equations using Hamilton's principle, and we show how his potential vorticity law arises from the usual particle-relabelling symmetry of the Lagrangian.

Although Miller and White wrote their equations for an ideal gas, it is illuminating to consider an ideal fluid with an *arbitrary* equation of state. We begin by recalling Hamilton's principle for such a fluid. Let

$$x(a, b, \tau), \qquad z(a, b, \tau) \tag{1}$$

be the horizontal and vertical location of the fluid particle identified by labelling coordinates (a, b) at time τ. (To simplify the equations, we temporarily assume that the fluid is two-dimensional and nonrotating.) The labels are assigned so that

$$\mathrm{d}a\,\mathrm{d}b = \mathrm{d}(mass). \tag{2}$$

Then

$$\alpha \equiv \frac{\partial(x, z)}{\partial(a, b)} \equiv \frac{1}{\rho} \tag{3}$$

is the specific volume, and the material derivative $(\partial/\partial\tau)$ of (3) yields the continuity equation

$$\frac{\partial \alpha}{\partial \tau} = \alpha\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \tag{4}$$

where $(u, w) = (\partial x/\partial\tau, \partial z/\partial\tau)$ is the fluid velocity. The momentum equations result from Hamilton's principle in the form

$$\delta\int\int\int \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}\tau\left\{\frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{\partial z}{\partial \tau}\right)^2 - E(\alpha, \theta) - gz\right\} = 0 \tag{5}$$

for arbitrary variations $\delta x(a, b, \tau)$, $\delta z(a, b, \tau)$ in the locations of the marked fluid particles. Here, g is the gravity constant, and the internal energy, $E(\alpha, \theta)$, is a *prescribed* function of the specific volume (3) and the potential temperature (or entropy),

$$\theta = \theta(a, b). \tag{6}$$

* Corresponding author: Scripps Institute of Oceanography, University of California, 9500 Gilman Drive, La Jolla, California 92093-0225 USA.

The potential temperature depends only on the particle identity, in a manner determined by initial conditions. For complete details, see, for example, Salmon (1988).

Now, if the contribution of the vertical velocity $\partial z/\partial\tau$ to the kinetic energy in (5) is simply discarded, then the resulting form of Hamilton's principle

$$\delta\iiint \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}\tau\left\{\frac{1}{2}\left(\frac{\partial x}{\partial\tau}\right)^2 - E(\alpha,\theta) - gz\right\} = 0 \tag{7}$$

yields the equations for hydrostatic flow:

$$\begin{aligned} \delta x\text{:}\quad & \frac{\partial^2 x}{\partial\tau^2} = -\alpha\frac{\partial p}{\partial x} \\ \delta z\text{:}\quad & 0 = -\alpha\frac{\partial p}{\partial z} - g \end{aligned} \tag{8}$$

where

$$p \equiv -\frac{\partial E(\alpha,\theta)}{\partial\alpha} \tag{9}$$

is the usual thermodynamic definition of pressure. By (6),

$$\frac{\partial\theta}{\partial\tau} = 0. \tag{10}$$

Equations (4, 8, 9, 10) are the complete set of equations for hydrostatic flow; (10) is the 'thermodynamic equation', and (9) can be considered the equation of state.

We begin by asserting that the variational principle (7) for the hydrostatic flow is equivalent to

$$\delta\iiint \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}\tau\left\{\frac{1}{2}\left(\frac{\partial x}{\partial\tau}\right)^2 - G(p,\theta) - \frac{\phi}{g}\left(\frac{\partial(x,p)}{\partial(a,b)} + g\right)\right\} = 0 \tag{11}$$

for variations $\delta x(a,b,\tau)$, $\delta p(a,b,\tau)$, $\delta\phi(a,b,\tau)$. Here,

$$G(p,\theta) \equiv E(\alpha,\theta) + p\alpha \tag{12}$$

is the free energy, a prescribed function of the pressure and potential temperature, determined from $E(\alpha,\theta)$ by (9). By (9) and (12),

$$\frac{\partial G(p,\theta)}{\partial p} = \alpha. \tag{13}$$

In the new variational principle (11), (x,p) play a role analagous to the role played by (x,z) in (7). Thus, by the device of introducing the free energy, $G(p,\theta)$, the variational principle (11) gives p the status of a generalized coordinate.

The form of (11) is logical, because

$$\iint \mathrm{d}x\,\mathrm{d}z\,\rho(E+gz) = \iint \mathrm{d}x\,\mathrm{d}z\,\rho G \tag{14}$$

in hydrostatic flow. The last term in (11) can be viewed as a constraint—with Lagrange multiplier ϕ—that p be hydrostatic. (It is this constraint that filters out acoustic waves.) However, it suffices to show that the equations resulting from (11) are equivalent to (4, 8–10). The variations of (11) yield:

$$\delta x\text{:}\quad -\frac{\partial^2 x}{\partial\tau^2} + \frac{1}{g}\frac{\partial(x,p)}{\partial(a,b)}\frac{\partial\phi}{\partial x}\Big)_p = 0 \tag{15}$$

$$\delta p\text{:}\quad -\frac{\partial G}{\partial p} + \frac{1}{g}\frac{\partial(x,p)}{\partial(a,b)}\frac{\partial\phi}{\partial p} = 0 \tag{16}$$

$$\delta\phi\text{:}\quad \frac{\partial(x,p)}{\partial(a,b)} + g = 0. \tag{17}$$

By (17), (15) and (16) can be rewritten in the familiar forms

$$\frac{\partial u}{\partial \tau} = -\frac{\partial \phi}{\partial x}\Big)_p \tag{18}$$

and

$$0 = -\frac{\partial \phi}{\partial p} - \alpha. \tag{19}$$

The material derivative of (17) is

$$\frac{\partial u}{\partial x}\Big)_p + \frac{\partial \omega}{\partial p} = 0 \tag{20}$$

where $\omega \equiv \partial p/\partial \tau$ is the vertical velocity in pressure coordinates. Thus (10, 13, 18–20) form a complete set of equations in (x, p) coordinates, equivalent to (4, 8–10) in (x, z) coordinates.

Now the hydrostatic Lagrangian in (7) and its equivalent in (11) differ from the exact Lagrangian in (5) only in that (5) includes the vertical kinetic energy. It follows that we can obtain more accurate approximations than (7) or (11) by including approximations to this vertical kinetic energy. The most natural such approximation relates the vertical velocity to the material derivative of the hydrostatic pressure p:

$$w \equiv \frac{\partial z}{\partial \tau} \approx \frac{\partial z}{\partial p}\frac{\partial p}{\partial \tau} \approx -\frac{\alpha}{g}\frac{\partial p}{\partial \tau} = -\frac{1}{g}\frac{\partial G(p, \theta)}{\partial \tau}. \tag{21}$$

Since the approximation (21) depends on p, it is better to work from (11) (for which p is a generalized coordinate, subject to independent variations) than from (7). We will show that the variational principle

$$\delta \iiint da\, db\, d\tau \left\{ \frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \frac{1}{2}\left(\frac{1}{g}\frac{\partial G(p, \theta)}{\partial \tau}\right)^2 - G(p, \theta) - \frac{\phi}{g}\left(\frac{\partial(x, p)}{\partial(a, b)} + g\right)\right\} = 0 \tag{22}$$

is equivalent to White's extended equations. The conservation laws for energy and potential vorticity stated by White are then automatic consequences of the obvious symmetry properties of (22).

Since the new (vertical velocity) term in (22) is affected only by $\delta p(a, b, \tau)$ variations, (18) and (20) are unchanged. The new vertical momentum equation is

$$\delta p: \quad -\frac{1}{g^2}\frac{\partial G}{\partial p}\frac{\partial^2 G}{\partial \tau^2} - \frac{\partial G}{\partial p} + \frac{1}{g}\frac{\partial(x, p)}{\partial(a, b)}\frac{\partial \phi}{\partial p} = 0. \tag{23}$$

Using (17) again, and remembering (13), (23) becomes

$$\frac{\alpha}{g^2}\frac{\partial}{\partial \tau}(\alpha\omega) = -\frac{\partial \phi}{\partial p} - \alpha. \tag{24}$$

For an ideal gas, $\alpha = RT/p$ and (10, 13, 18, 20, 24) reduce to the equations discovered by White.

The dynamics resulting from (22) automatically conserves the energy

$$\iint da\, db \left\{ \tfrac{1}{2}u^2 + \frac{1}{2}\left(\frac{\alpha}{g}\omega\right)^2 + G(p, \theta)\right\}. \tag{25}$$

For an ideal gas, $G = C_pT$ and (25) reduces to the conserved energy given by White.

Before discussing potential vorticity, we generalize (22) to include Coriolis force and the other horizontal dimension (y). Let c be the third labelling coordinate, again defined so that

$$\alpha = \frac{\partial(x, y, z)}{\partial(a, b, c)}. \tag{26}$$

It is convenient to let $c = \theta$. Let $R(x, y)$ and $P(x, y)$ be any two functions satisfying

$$\frac{\partial R}{\partial y} + \frac{\partial P}{\partial x} = f(x, y) \tag{27}$$

where f is the Coriolis parameter. Then the variational principle

$$\delta\int d\tau\iiint da\,db\,dc\left\{\left(\frac{1}{2}\frac{\partial x}{\partial\tau}-R(x,y)\right)\frac{\partial x}{\partial\tau}+\left(\frac{1}{2}\frac{\partial y}{\partial\tau}+P(x,y)\right)\frac{\partial y}{\partial\tau}+\frac{1}{2}\left(\frac{1}{g}\frac{\partial G(p,\theta)}{\partial\tau}\right)^2-\right.$$
$$\left.-G(p,\theta)-\frac{\phi}{g}\left(\frac{\partial(x,y,p)}{\partial(a,b,c)}+g\right)\right\}=0 \tag{28}$$

yields

$$\delta x,\ \delta y:\quad \frac{\partial u}{\partial\tau}-fv=-\frac{\partial\phi}{\partial x}\Big)_p \quad\text{and}\quad \frac{\partial v}{\partial\tau}+fu=-\frac{\partial\phi}{\partial y}\Big)_p. \tag{29}$$

Equations (10, 13, 24) are unchanged, and (20) generalizes in the obvious way.

Now, Hamilton's principle (28) applies to time-dependent variations in the map from (a, b, c) to (x, y, p). But consider the inverse map from (x, y, p) to (a, b, c). Remembering that $c = \theta$, we see that only the $\partial/\partial\tau$ terms in (28) are affected by variations δa, δb, $\delta c(x, y, p, t)$ satisfying

$$\delta\frac{\partial(a,b,c)}{\partial(x,y,p)}=0 \quad\text{and}\quad \delta c=0. \tag{30}$$

But (30) implies that

$$\frac{\partial\delta a}{\partial a}+\frac{\partial\delta b}{\partial b}=0 \tag{31}$$

so that

$$\delta a=-\frac{\partial}{\partial b}\delta\psi(a,b,c,\tau),\qquad \delta b=\frac{\partial}{\partial a}\delta\psi(a,b,c,\tau),\qquad \delta c=0 \tag{32}$$

for some $\delta\psi(a, b, c, \tau)$. Now, it is easy to show that, for any quantity, F, the variation in material derivative $\partial F/\partial\tau$ caused by particle-label variations satisfying (30) is:

$$\delta\frac{\partial F}{\partial\tau}=-\frac{\partial F}{\partial a}\frac{\partial}{\partial\tau}\delta a-\frac{\partial F}{\partial b}\frac{\partial}{\partial\tau}\partial b. \tag{33}$$

Substituting (33) and (32) into (28), and integrating by parts, we obtain

$$\int d\tau\iiint da\,db\,dc\,\frac{\partial Q}{\partial\tau}\delta\psi=0 \tag{34}$$

where

$$Q=\frac{\partial(x,u-R,\theta)}{\partial(a,b,\theta)}+\frac{\partial(y,v+P,\theta)}{\partial(a,b,\theta)}+\frac{1}{g^2}\frac{\partial(G,G_\tau,\theta)}{\partial(a,b,\theta)}. \tag{35}$$

Since $\delta\psi$ is arbitrary, $\partial Q/\partial\tau=0$. Multiplying (35) by the constant

$$\frac{\partial(a,b,\theta)}{\partial(x,y,p)} \tag{36}$$

(see (17) and remember $c = \theta$) and noting that

$$\frac{\partial(G,G_\tau,\theta)}{\partial(x,y,p)}=\alpha\frac{\partial(p,\alpha p_\tau,\theta)}{\partial(x,y,p)}=\frac{\partial(p,\alpha^2\omega,\theta)}{\partial(x,y,p)} \tag{37}$$

because, by (13), $\alpha = \alpha(p, \theta)$, we finally conclude that

$$\frac{\partial}{\partial\tau}\left\{\frac{\partial(x,u-R,\theta)}{\partial(x,y,p)}+\frac{\partial(y,v+P,\theta)}{\partial(x,y,p)}+\frac{\partial(p,\alpha^2\omega/g^2,\theta)}{\partial(x,y,p)}\right\}$$
$$=\frac{\partial}{\partial\tau}[\{\nabla_p\times(u-R,v+P,\alpha^2\omega/g^2)\}\cdot\nabla_p\theta]=0 \tag{38}$$

where ∇_p is the gradient operator in (x, y, p) coordinates. Equation (38) is the potential vorticity law discovered by White.

Although our derivation of White's equations from (28) adds nothing of practical value to the beautiful results of Miller, Pearce and White, it offers another illustration of the great compactness and power of Hamiltonian approximation methods. The existence of energy and potential-vorticity conservation laws is guaranteed by the obvious symmetry properties of (28), and even the absence of sound waves could be anticipated from the fact the pressure inside the free energy $G(p, \theta)$ in (28) is hydrostatic (by the ϕ-constraint in (28)) and thus G is insensitive to local pressure fluctuations. Moreover, in the Hamiltonian derivation, one is not tempted to introduce the reference state represented by $T_s(p)$ that seems to arise when the approximate equations are obtained by the more conventional method of expanding in a small parameter. In that respect, White's equations are analogous to the semigeostrophic equations for nearly geostrophic flow, which do not incorporate a prescribed reference state, while the earlier equations of Miller and Pearce are analogous to the quasigeostrophic equations, which become highly inaccurate when the flow wanders far from a prescribed state with nearly flat isothermal surfaces.

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