Problem Set 5: Solutions

1. (10 pts) A matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \\ 2 & 2 \end{bmatrix}$$

Using the eigenvector relations between the singular values, $\mathbf{u}_i$, and $\mathbf{v}_i$, find the complete SVD for $\mathbf{A}$. Do not use a singular value decomposition program.

Solution: Although you could obtain the solution to this problem by plugging your results into Matlab's SVD package, that is not the point of this exercise. Instead follow the procedure that we used in class to derive the SVD:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 14 & 4 \\ 4 & 14 \end{bmatrix}$$

Now we solve the eigenvalue equation:

$$\mathbf{A}^T\mathbf{A}\mathbf{v}_i = \lambda_i^2\mathbf{v}_i$$

In order to find a nontrivial solution for $\mathbf{v}_i$, we require that the determinant of the matrix $\mathbf{A}^T\mathbf{A} - \lambda_i^2\mathbf{I}$ be zero:

$$(14 - \lambda_i^2)^2 - 16 = \lambda_i^4 - 28\lambda_i^2 + 180 = (\lambda_i^2 - 10)(\lambda_i^2 - 18) = 0$$

Therefore $\lambda_1^2 = 18$ and $\lambda_2^2 = 10$, and the singular values are $\sqrt{18}$ and $\sqrt{10}$.

Now we find the singular vectors $\mathbf{v}_i$ that correspond to each λ_i^2. For λ_1:

$$\mathbf{A}^T\mathbf{A} - \lambda_1^2\mathbf{I} = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}$$

So $\mathbf{v}_1^T = 1/\sqrt{2}[1 \;\; 1]$. For λ_2:

$$\mathbf{A}^T\mathbf{A} - \lambda_2^2\mathbf{I} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

So $\mathbf{v}_2^T = 1/\sqrt{2}[-1 \;\; 1]$, which is orthogonal to $\mathbf{v}_1$.

Finally we need the singular vectors $\mathbf{u}_i$. For this we can compute $\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{u}_\mathbf{i}$. Therefore

$$\mathbf{u}_1 = \frac{1}{\sqrt{18}}\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sqrt{10}}\mathbf{A}\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}.$$

The final singular vector $\mathbf{u}_3$ must be orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. This requirement can be met using Gram-Schmidt orthogonalization or by inspection. The result is $\mathbf{u}_3^T = 1/\sqrt{45}[4\ \ -2\ \ -5]$

2. (20 pts) Suppose we have observations of horizontal velocity from a moored array of current meters, as described in the following table:

x (km)	y (km)	u (cm/s)	v (cm/s)
10.	0.	35.5	21.3
0.	10.	53.5	30.4
-10.	0.	49.2	24.8
0.	-10.	33.5	18.4
0.	0.	43.5	25.5

A. Fit planes to the measurements of u and v and calculate the following:

i. The coefficient of each function in the fit.

ii. The mapped values of velocity at each instrument.

iii. The misfit as measured by the L2 norm.

iv. The vorticity, $\partial v/\partial x - \partial u/\partial y$, and divergence, $\partial u/\partial x + \partial v/\partial y$.

v. The covariance matrix of the model coefficients, assuming that the standard deviation of each component of velocity is 3.0 cm/s and that the error is uncorrelated between instruments.

vi. The resulting error in area-averaged velocity, vorticity, and divergence.

B. Enforce the constraint that the flow has zero divergence, and redo i-vi above. Discuss the differences between the two sets of results.

C. Suppose one of the instruments failed. (You choose which one.) Redo the calculations and discuss the results.

Solution:
A. Begin by defining a matrix $\mathbf{G}$:

$$\mathbf{G} = \begin{bmatrix} 1 & 10 & 0 \\ 1 & 0 & 10 \\ 1 & -10 & 0 \\ 1 & 0 & -10 \\ 1 & 0 & 0 \end{bmatrix}.$$

Now, a planar fit to the u velocities is $\mathbf{m} = (\mathbf{G^TG})^{-1}\mathbf{G^Tu}$, and the same thing can be done for the v velocities. Thus:

$$\begin{aligned} u &= 43.0400 - 0.6850x + 1.0000y \\ v &= 24.0800 - 0.1750x + 0.6000y \end{aligned}$$

where constant coefficients are in cm s^{-1} and coefficients in front of x and y are in cm s^{-1} km^{-1}. At each instrument, the least-squares fit produces the following values:

$$\mathbf{u}_{fit} = \begin{bmatrix} 36.1900 \\ 53.0400 \\ 49.8900 \\ 33.0400 \\ 43.0400 \end{bmatrix} \qquad \mathbf{v}_{fit} = \begin{bmatrix} 22.3300 \\ 30.0800 \\ 25.8300 \\ 18.0800 \\ 24.0800 \end{bmatrix}$$

The L2 norm of the misfit is 1.2598 for u and 2.0840 for v.
The vorticity and divergence can be found from the fitted coefficients, which are in cm s^{-1} km^{-1} or 10^{-5} s^{-1}.

$$\begin{aligned} \text{vorticity} &= m_v(2) - m_u(3) = -1.1750 \\ \text{divergence} &= m_u(2) + m_v(3) = -0.0850 \end{aligned}$$

The covariance matrix $\mathbf{C}$ is $3^2(\mathbf{G^T G})^{-\mathbf{1}}$.

$$\mathbf{C} = \begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 4.5 \times 10^{-2} & 0 \\ 0 & 0 & 4.5 \times 10^{-2} \end{bmatrix}.$$

The velocity covariance can be computed as $\mathbf{GCG^T}$. So the area average velocity variance (for u or v) is the sum over all these points, divided by N^2. Therefore $\sigma = 1.3$cm s^{-1}. The vorticity and divergence errors are determined from the fitted coefficients: $\sigma = \sqrt{C_{2,2} + C_{3,3}} = 0.3 \times 10^{-5}$ s^{-1}.

B. To enforce a non-divergence constraint, we require that $m_u(2) = -m_v(3)$. This couples the least-squares equations for u and v together, so we define a new matrix $\mathbf{G}$ with 10 rows (corresponding to the u and v measurements) and 5 unknowns. The new solution is

$$\begin{aligned} u &= 43.0400 - 0.6425x + 1.0000y \\ v &= 24.0800 - 0.1750x + 0.6425y \end{aligned}$$

Fitted values at the instrument locations are:

$$\mathbf{u}_{fit} = \begin{bmatrix} 36.6150 \\ 53.0400 \\ 49.4650 \\ 33.0400 \\ 43.0400 \end{bmatrix} \qquad \mathbf{v}_{fit} = \begin{bmatrix} 22.3300 \\ 30.5050 \\ 25.8300 \\ 17.6550 \\ 24.0800 \end{bmatrix}$$

The L2 norm of the misfit is 2.5792 for the combined solution.

The vorticity and divergence can be found once again:

$$\begin{aligned} \text{vorticity} &= -1.1750 \\ \text{divergence} &= 0.0 \end{aligned}$$

The covariance matrix is $\mathbf{C} = 3(\mathbf{G^T G})^{-\mathbf{1}}$ and is once again diagonal.

$$\mathrm{diag}(\mathbf{C}) = \begin{bmatrix} 1.8 & 2.2 \times 10^{-2} & 4.5 \times 10^{-2} & 1.8 & 4.5 \times 10^{-2} \end{bmatrix}.$$

This leads to an area average velocity error of 1.9, a divergence error of 2.1×10^{-5} s^{-1}, and a vorticity error of 3×10^{-5} s^{-1}. The assumption of zero divergence decreases the apparent divergence error, since more observations are available to constrain the result. It does not change the vorticity error, and it slightly increases the total velocity uncertainty.

C. Results of this section depend on the choice of instrument to remove. Overall, you can expect that uncertainties will increase, particularly at the location without observations.

3. (bonus) Choose any oceanographic or climate record that you like. Least-squares fit a function of your choice to the data. Discuss the method of fitting that you have used, uncertainties in the results, and the goodness of fit. If you e-mail me a graphic summarizing your results (preferably in postscript, pdf, gif, or jpg) I'll bring them to class for discussion and will post them on the web.

Evaluation criteria: Have you explained the origin of your data, your choice of fitting functions and methodology, the uncertainties in your results, and the misfit of the final result?