

Lecture 18:

Recap

Last time we looked at the transfer function, which we framed as the Fourier transformed form of a regression coefficient. Today we'll pick up a few final details, looking at some examples and at the impact of noise on our coherence and transfer function estimates.

Transfer function

Last time we wrote the transfer function H in the form such that, if

$$y_t = \int_{-\infty}^{\infty} h(u)z(t-u) du, \quad (1)$$

the Fourier transform is:

$$Y(f) = H(f)Z(f). \quad (2)$$

There are some nice rules for transfer functions:

1. **Linearity:** If a given linear system has an input $z_1(t)$ which corresponds to an output $y_1(t)$, and input $z_2(t)$ corresponds to output $y_2(t)$ then a summed input $z(t) = \alpha z_1(t) + \beta z_2(t)$, will produce an output $y(t) = \alpha y_1(t) + \beta y_2(t)$.
2. **Time invariance:** If an input is delayed in time by τ , then the output is as well: If $z(t) \rightarrow z(t + \tau)$, then $y(t) \rightarrow y(t + \tau)$.
3. **Causality:** If $h(t)$ represents an impulse, then it should be zero for $t < 0$. A response cannot occur before the forcing.
4. **Sequential application:** If the output of one linear system is an input to a second system, then the frequency response is

$$H_{12}(f) = H_1(f) \cdot H_2(f) \quad (3)$$

So suppose we measure $y(t)$ and $z(t)$. Can we determine h or H ? We know that

$$Y(f) = H(f)Z(f) \quad (4)$$

Let's multiply both sides of the equation by the complex conjugate of Z to form the cross-spectrum:

$$\frac{Y(f)Z^*(f)}{N^2 \Delta f} = H(f) \frac{Z(f)Z^*(f)}{N^2 \Delta f} \quad (5)$$

This becomes

$$G_{zy}(f) = H(f)G_{zz}(f) \quad (6)$$

so

$$H(f) = \frac{G_{zy}(f)}{G_{zz}(f)} \quad (7)$$

Salinity spiking examples

A classic example comes from the 'salinity spiking' problem, which arises due to the different response times of temperature and conductivity sensors:

$$S = S(T(t), C(t - \tau)). \quad (8)$$

Our challenge now is to use the transfer (or gain) function to assess the salinity spiking, and maybe even to correct it. Conductivity is strongly dependent on temperature, and we have to remove the temperature effect to determine salinity. But the faster response times of conductivity sensors relative to temperature sensors are a source of confusion. Here's a basic procedure.

1. Identify a segment of the water column in which temperature and conductivity should be well behaved, with fluctuations due to temperature only. (This isn't essential, but it will give us a good shot at unraveling the sensor response time issues that lead to salinity or density spiking.) We'll identify the true values as T and C , and the measured values as \hat{T} and \hat{C} .
2. Collect a lot of profiles of data.
3. Now treat this as a linear system:

$$\hat{T}(k) = H_T(k)T(k) \quad (9)$$

$$\hat{C}(k) = H_C(k)T(k) \quad (10)$$

$$(11)$$

Here $H_T(k)$ is the spatial/frequency response of the temperature sensor, and $H_C(k)$ is the spatial/frequency response of the conductivity sensor. The use of $T(k)$ in the conductivity equation might seem a little crazy, but it's really important, since we're asserting that salinity is unimportant in our (hypothetical) study region.

4. Compute cross spectra:

$$\hat{G}_{TT}(k) = \frac{\langle \hat{T}^*(k)\hat{T}(k) \rangle}{N^2 \Delta k} \quad (12)$$

$$\hat{G}_{CC}(k) = \frac{\langle \hat{C}^*(k)\hat{C}(k) \rangle}{N^2 \Delta k} \quad (13)$$

$$\hat{G}_{TC}(k) = \frac{\langle \hat{T}^*(k)\hat{C}(k) \rangle}{N^2 \Delta k} \quad (14)$$

$$(15)$$

Then if we substitute in the expressions linking the observed values to the true values we obtain:

$$\frac{\langle \hat{T}^*(k)\hat{C}(k) \rangle}{N\Delta t} = \frac{\langle (H_T(k)T(k))^* H_C(k)T(k) \rangle}{N^2 \Delta k} \quad (16)$$

$$= [H_T^*(k)H_C(k)] \frac{\langle T^*(k)T(k) \rangle}{N^2 \Delta k} \quad (17)$$

$$\hat{G}_{TC}(k) = [H_T^*(k)H_C(k)]G_{TT}(k) \quad (18)$$

The same applies for the temperature spectrum:

$$\hat{G}_{TT}(k) = \frac{\langle \hat{T}^*(k)\hat{T}(k) \rangle}{N^2 \Delta k} \quad (19)$$

$$= [H_T^*(k)H_T(k)]G_{TT}(k) \quad (20)$$

So the ratio of these becomes:

$$\frac{\hat{G}_{TC}(k)}{\hat{G}_{TT}(k)} = \frac{H_T^*(k)H_C(k)}{|H_T|^2} = \frac{H_C(k)}{H_T(k)} \quad (21)$$

This means that even without knowing the response function H , we can compute the ratio of the response functions from the transfer function, the ratio of the cross-spectrum to the spectrum.

An analogous relationship also holds:

$$\frac{\hat{G}_{TC}(k)}{\hat{G}_{CC}(k)} = \frac{H_T^*(k)H_C(k)}{|H_C|^2} = \frac{H_T(k)}{H_C(k)} \quad (22)$$

5. Now use this information to correct the conductivity sensor to have the same response as the temperature sensor. Here we'll define our corrected conductivity as $\hat{\hat{C}}(k)$, and we want to understand its relationship with the observed temperature $\hat{T}(k)$ and the true temperature $T(k)$.

$$\hat{\hat{C}}(k) = \alpha \hat{T}(k) = \alpha H_T(k)T(k) \quad (23)$$

This means we need a correction of the form:

$$\hat{\hat{C}}(k) = \hat{C}(k) \cdot P(k) = \alpha \hat{T}(k), \quad (24)$$

and our task is to figure out $P(k)$. We can also write:

$$\hat{C}(k) \cdot P(k) = H_C(k)T(k)P(k) \quad (25)$$

so putting this together:

$$\alpha H_T(k)T(k) = H_C(k)T(k)P(k) \quad (26)$$

Thus

$$P(k) = \frac{\alpha H_T(k)}{H_C(k)} = \alpha \frac{\hat{G}_{TC}(k)}{\hat{G}_{CC}(k)} \quad (27)$$

where α is real. So we do a bit of curve fitting to optimize our correction.

A typical correction might allow for errors both in the response time and a direct time lag:

$$\tau \frac{d\hat{T}(t)}{dt} + \hat{T}(t) = T(t - L) \quad (28)$$

(from Giles and McDougall, Deep-Sea Research, 1986) and this suggests corrections both in the frequency and time domain, either by minimizing phase differences or by maximizing correlation. We can Fourier transform this to find:

$$-i2\pi f\tau \mathcal{F}(\hat{T}) + \mathcal{F}(\hat{T}) = \mathcal{F}(T)e^{-i2\pi fL}, \quad (29)$$

implying that

$$(1 - i2\pi f\tau)e^{i2\pi fL} \mathcal{F}(\hat{T}) = \mathcal{F}(T). \quad (30)$$

Since conductivity has the fast response, one strategy is to treat $\hat{\hat{C}}$ as behaving like the true temperature T . So hypothetically:

$$\hat{T}(k) = H_T(k)T(k) = \frac{1}{(1 - i2\pi f\tau)e^{i2\pi fL}} T(k) \quad (31)$$

$$\hat{C}(k) = H_C(k)T(k) = T(k) \quad (32)$$

$$(33)$$

implying a correction $P(k)$ of the form:

$$P(k) = \frac{\alpha H_T(k)}{H_C(k)} = \alpha \frac{1}{(1 - i2\pi f\tau)e^{i2\pi fL}}. \quad (34)$$

Transfer function: Practical implication

You might be wondering exactly how to compute a transfer function, so here's a practical example.

```
lambda=10; % 10 m wavelength
V=0.3; % 0.3 m/s propagation
n2s=0.2; % noise-to-signal ratio
time=(1:5000)';
x=n2s*randn(5000,1)+cos(2*pi/lambda*V*time);
y=n2s*randn(5000,1)+cos(2*pi/lambda*V*(time)+pi/2);

% start with a segmented data approach:
xx=[reshape(x,500,10) reshape(x(251:4750),500,9)];
yy=[reshape(y,500,10) reshape(y(251:4750),500,9)];
fxx=fft(xx.*(hanning(500)*ones(1,19)));
fyy=fft(yy.*(hanning(500)*ones(1,19)));
sxx=abs(fxx(1:251,:)).^2;
syy=abs(fyy(1:251,:)).^2;
sxy=conj(fxx(1:251,:)).*fyy(1:251,:);
coher1=abs(mean(sxy,2)).^2./(mean(sxx,2).*mean(syy,2));
phase1=atan2(-imag(mean(sxy,2)),real(mean(sxy,2)));
semilogx(0:250,coher1)
transfer=mean(sxy,2) ./mean(sxx,2);
semilogx(0:250,coher1,0:250,abs(transfer))
set(gca,'FontSize',14)
xlabel('frequency','FontSize',14)
ylabel('coherence or transfer function','FontSize',14)
legend('coherence','transfer function amplitude')
```

In this case, you'll see that the transfer function peak happens to be about the same size as the coherence, while the background noise for the transfer function is significantly larger than the background noise for the coherence. These results are entirely a function of the data that we use. The high background noise level stems from the fact that the coherence is scaled by x , which can be very small.