Dissipative Descent: Rocking and Rolling down an incline

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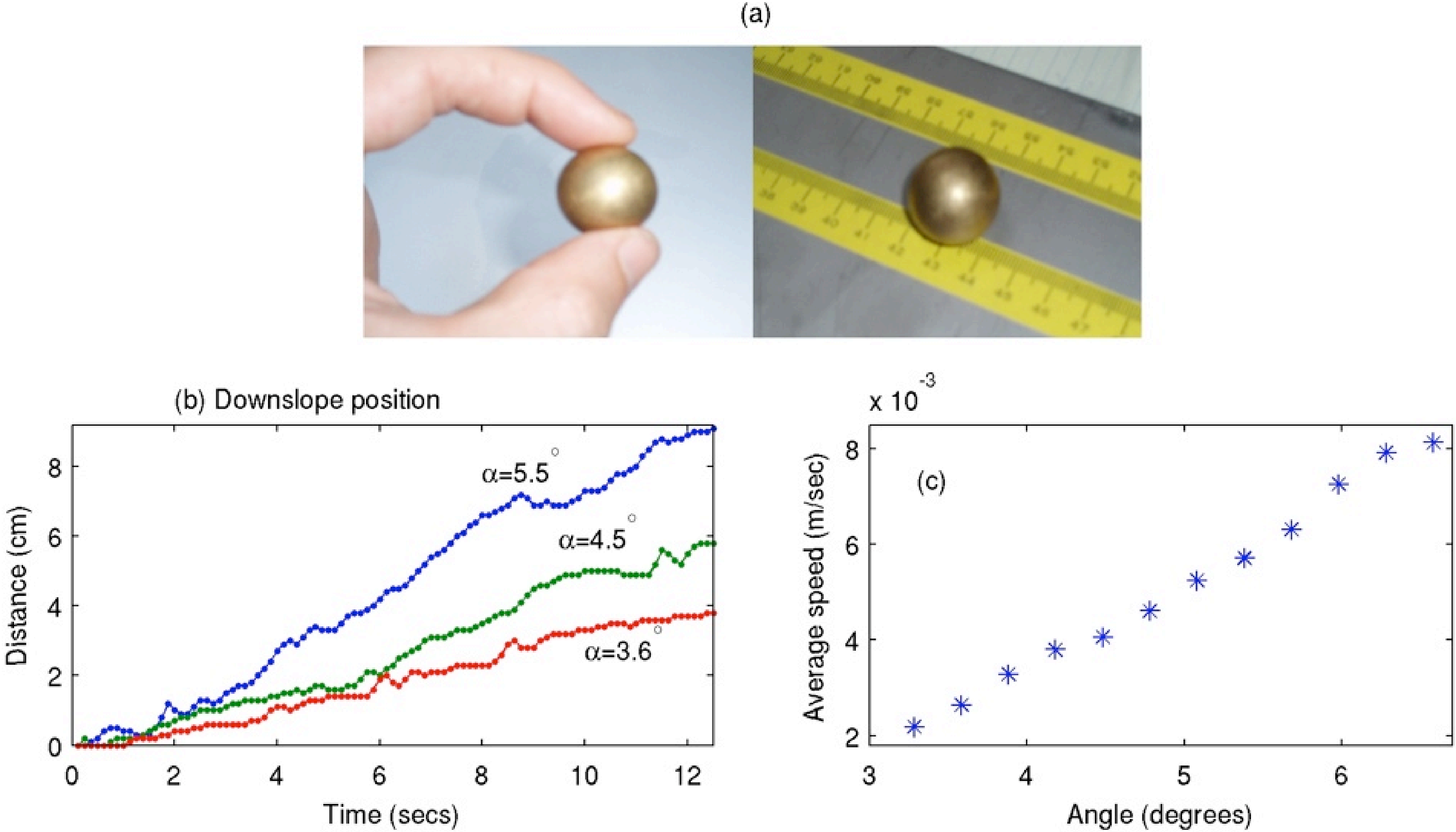
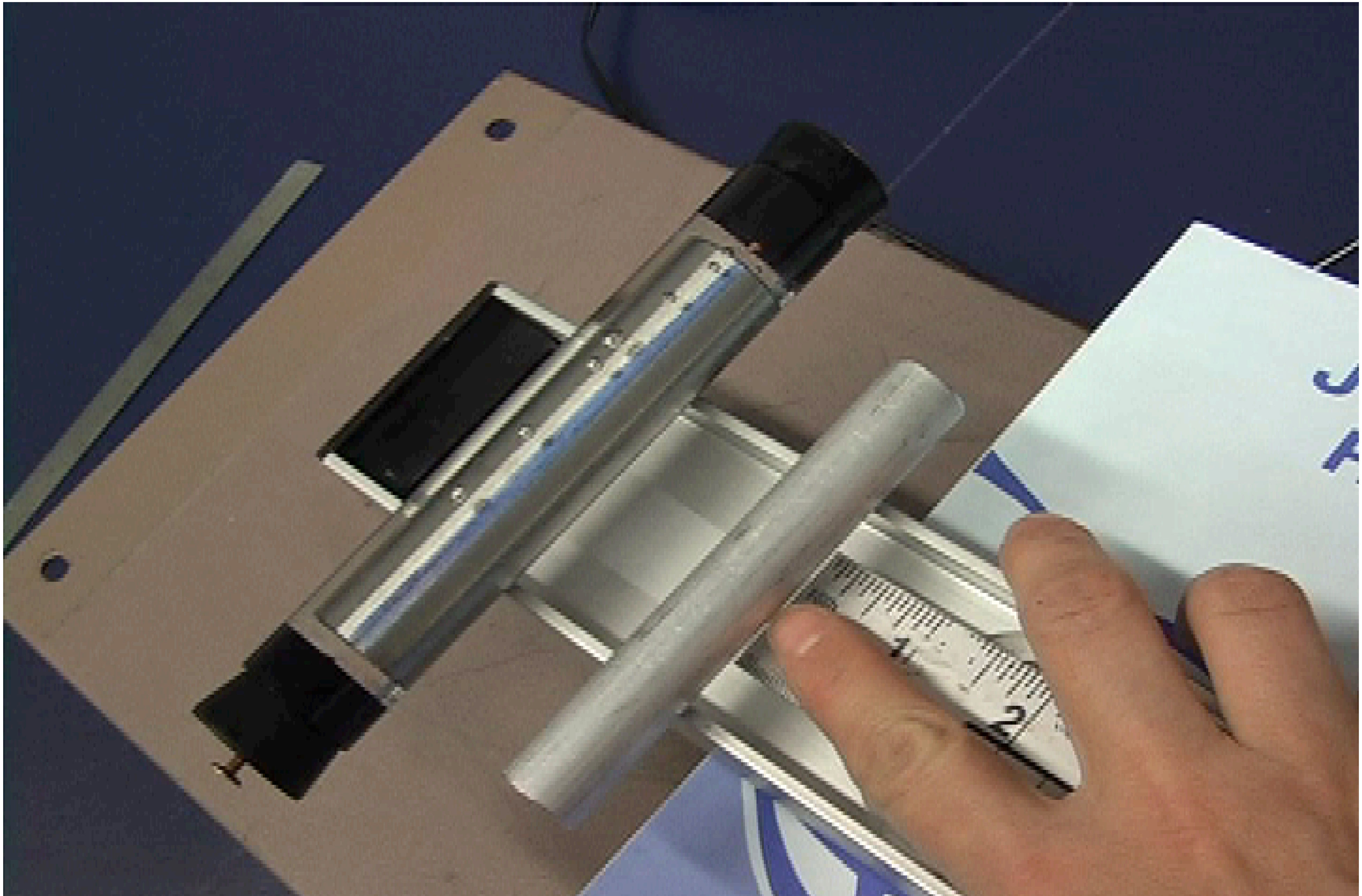
The snail ball from “Grand Illusions” - $50



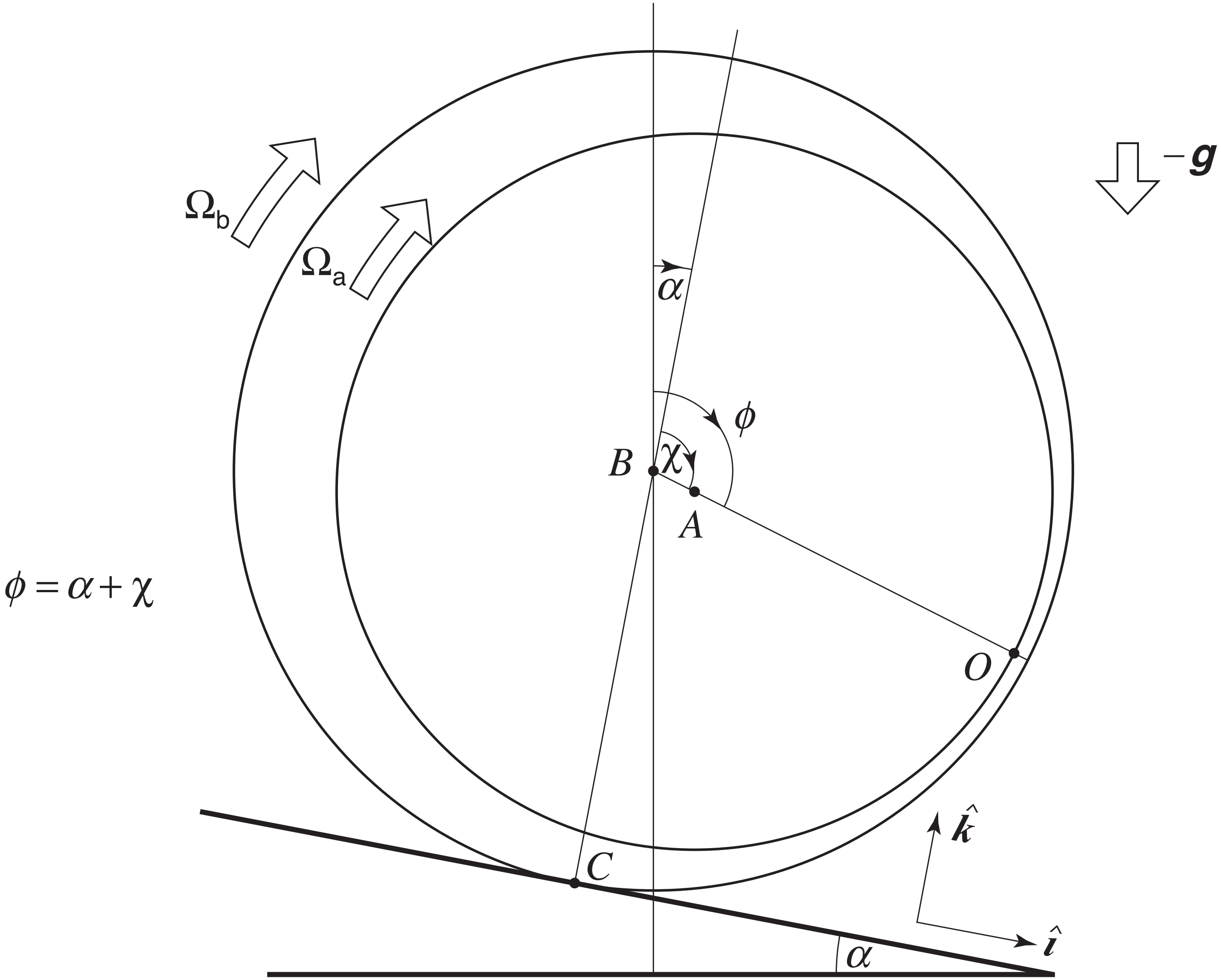
"A small metallic gold ball just over 2cm in diameter ... the ball does roll, but does so incredibly slowly. To an audience it seems baffling ... inside the ball, which is actually hollow, there is a viscous liquid and a smaller ball which is heavy... it is the smaller heavier ball which determines the pace and this is slow because of the viscous liquid."

The Snail Cylinder from UBC - $10,000



The state of the snail cylinder is specified by

$$\epsilon(t)\,, \qquad \chi(t)\,, \qquad \Omega_{\mathsf{a}}(t)\,, \qquad \text{and} \qquad \Omega_{\mathsf{b}}(t)$$



The flow in the gap is analyzed with the lubrication approximation.

$$b - a = \delta \ll a$$

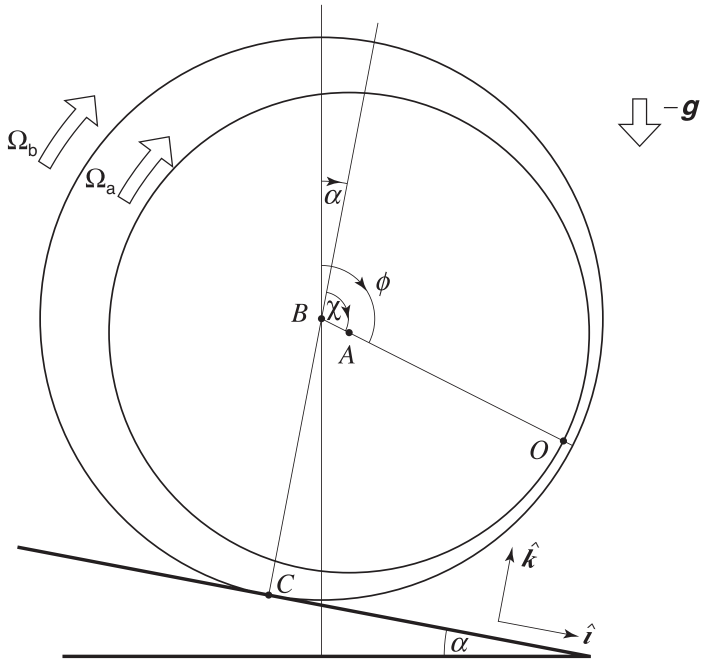
$$M \equiv m_{\mathsf{a}} + m_{\mathsf{b}} + m_{\mathsf{f}}$$

$$m'_{\mathsf{a}} \equiv m_{\mathsf{a}} - m''_{\mathsf{a}}$$

$$m''_{\mathsf{a}} \equiv \pi a^2 L \rho$$

$$\boldsymbol{X}_{\mathsf{a}} = \boldsymbol{X}_{\mathsf{b}} + \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} \equiv \epsilon\left(\sin\chi\,\hat{\boldsymbol{\imath}} + \cos\chi\,\hat{\boldsymbol{k}}\right)$$

The equations of motion



$$\tfrac{1}{2} m_a a^2 \dot{\Omega}_a = T_a$$

$$\left(m_a + \frac{m_a''^2}{m_f} \right) (\epsilon \ddot{\chi} + 2\dot{\epsilon}\dot{\chi}) = f_\chi + m_a' g \sin\phi - m_a' b \dot{\Omega}_b \cos\chi \,,$$

$$\left(m_a + \frac{m_a''^2}{m_f} \right) (\ddot{\epsilon} - \epsilon \dot{\chi}^2) = f_\epsilon - m_a' g \cos\phi - m_a' b \dot{\Omega}_b \sin\chi \,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\tfrac{1}{2} m_a a^2 \Omega_a + (M + m_b) b^2 \Omega_b + \left(m_a + \frac{m_a''^2}{m_f} \right) \epsilon^2 \dot{\chi} + m_a' b \frac{\mathrm{d}}{\mathrm{d}t} (\epsilon \sin\chi) \right]$$

$$+ m_a' b \dot{\Omega}_b \epsilon \cos\chi \approx M g b \sin\alpha + m_a' g \epsilon \sin\phi$$

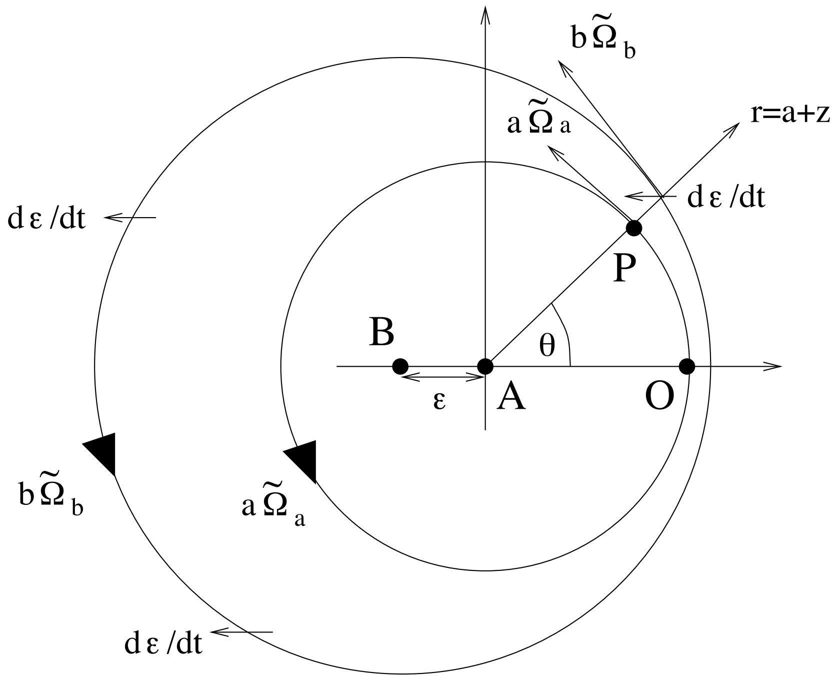
The only approximation (so far)

Recall:

$$\boldsymbol{X}_a = \boldsymbol{X}_b + \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} \equiv \epsilon \left(\sin\chi\, \hat{\boldsymbol{\imath}} + \cos\chi\, \hat{\boldsymbol{k}} \right)$$

The lubrication approximation in the gap expresses the hydrodynamic forces and torques in terms of the four independent variables.

Lubrication in the gap: use the c. of m. frame of the fluid.



$$h(\theta) = \delta - \epsilon \cos\theta$$

$$\rho\nu u_{zz} = a^{-1} p_\theta, \qquad p_z = 0,$$

$$\frac{1}{a} u_\theta + w_z = 0$$

The rotational part: $$u^{\mathsf{R}}(z,\theta) = \left(1 - \frac{z}{h}\right) a\tilde{\Omega}_{\mathsf{a}} + \frac{z}{h} a\tilde{\Omega}_{\mathsf{b}} - \frac{p_\theta^{\mathsf{R}}}{2a\rho\nu} z(h-z)$$

The squeeze part: $$u^{\mathsf{S}}(z,\theta) = \frac{z}{h}\dot{\epsilon}\sin\theta - \frac{p_\theta^{\mathsf{S}}}{2a\rho\nu} z(h-z)$$

The pressure follows from global mass conservation. Integration round the inner cylinder gives the forces and torques.

The full Monty

$$\left(m_{\mathsf{a}}+\frac{m_{\mathsf{a}}''^2}{m_{\mathsf{f}}}\right)(\epsilon\ddot{\chi}+2\dot{\epsilon}\dot{\chi}) = f_\chi + m_{\mathsf{a}}' g\sin\phi - m_{\mathsf{a}}' b\dot{\Omega}_{\mathsf{b}}\cos\chi\,,$$

$$\left(m_{\mathsf{a}}+\frac{m_{\mathsf{a}}''^2}{m_{\mathsf{f}}}\right)(\ddot{\epsilon}-\epsilon\dot{\chi}^2) = f_\epsilon - m_{\mathsf{a}}' g\cos\phi - m_{\mathsf{a}}' b\dot{\Omega}_{\mathsf{b}}\sin\chi\,,$$

$$\tfrac{1}{2}m_{\mathsf{a}}a^2\dot{\Omega}_{\mathsf{a}} = T_{\mathsf{a}}$$

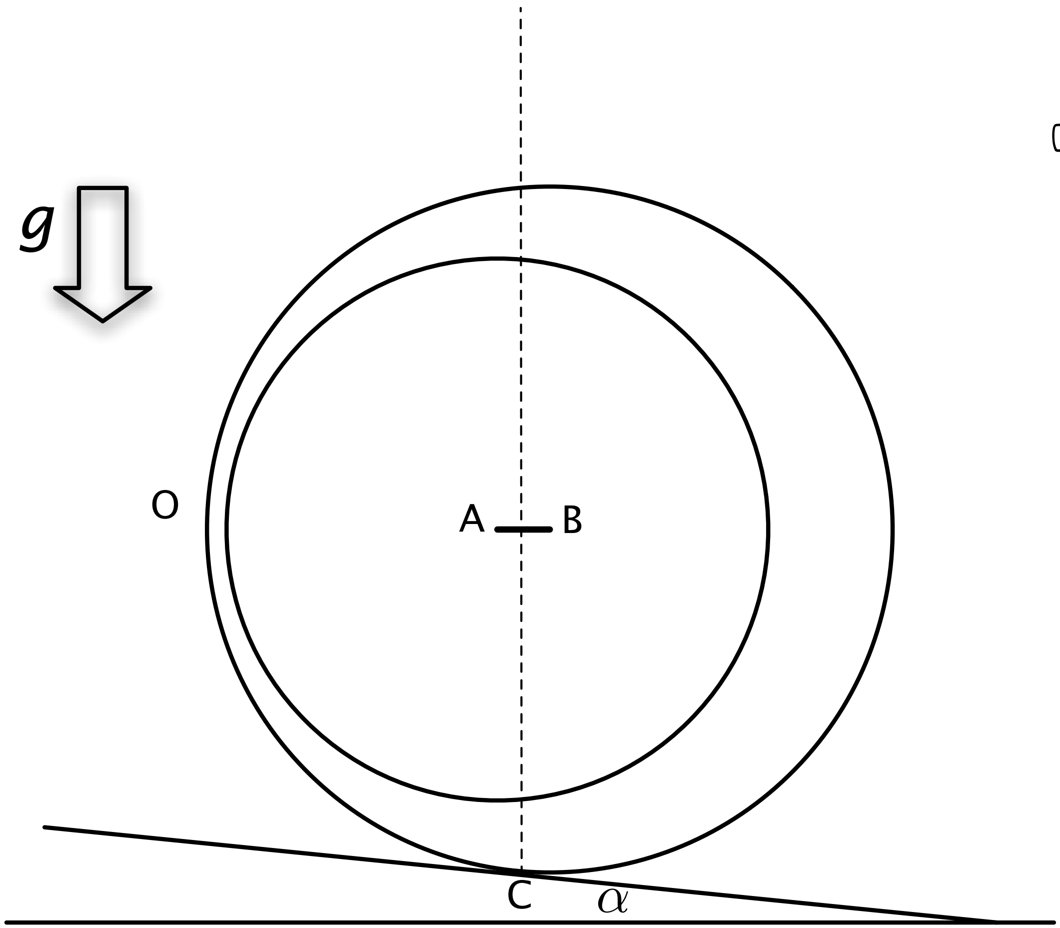
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}m_{\mathsf{a}}a^2\Omega_{\mathsf{a}} + (M+m_{\mathsf{b}})b^2\Omega_{\mathsf{b}} + \left(m_{\mathsf{a}}+\frac{m_{\mathsf{a}}''^2}{m_{\mathsf{f}}}\right)\epsilon^2\dot{\chi} + m_{\mathsf{a}}' b\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\sin\chi)\right]$$

$$+m_{\mathsf{a}}' b\dot{\Omega}_{\mathsf{b}}\epsilon\cos\chi \approx Mgb\sin\alpha + m_{\mathsf{a}}' g\epsilon\sin\phi$$

$$f_\epsilon = -\frac{12\nu a m_{\mathsf{a}}''}{\delta^2}\frac{\dot{\kappa}}{(1-\kappa^2)^{3/2}}\,, \qquad f_\chi = \frac{12\nu a m_{\mathsf{a}}''}{\delta^2}\frac{\kappa(\Omega_{\mathsf{a}}+\Omega_{\mathsf{b}}-2\dot{\chi})}{(2+\kappa^2)\sqrt{1-\kappa^2}}$$

$$T_{\mathsf{a}} = \frac{12\nu a m_{\mathsf{a}}''}{\delta}\frac{(1-\kappa^2)(\Omega_{\mathsf{b}}-\dot{\chi})-(1+2\kappa^2)(\Omega_{\mathsf{a}}-\dot{\chi})}{3(2+\kappa^2)\sqrt{1-\kappa^2}} \qquad \kappa(t)\equiv\frac{\epsilon(t)}{\delta}$$

A non-accelerating solution



The c. of m. lies directly above the point of contact and the line of centers is horizontal.

☞ $\dot{\Omega}_{\mathsf{a}} = \dot{\Omega}_{\mathsf{a}} = \dot{\epsilon} = \dot{\chi} = 0$

☞ $\Omega_{\mathsf{b}} \ \& \ \Omega_{\mathsf{a}} \propto \dfrac{1}{\sin\alpha}$
(this is crazy)

☞ We are relieved to discover that this solution is linearly unstable

The main approximation:

$$\frac{\delta}{a} \ll 1 \quad \text{and} \quad \sin\alpha \sim \frac{\delta}{a} \quad \Rightarrow$$

$$\left(m_{\mathsf{a}} + \frac{m_{\mathsf{a}}''^2}{m_{\mathsf{f}}}\right)(\epsilon\ddot{\chi} + 2\dot{\epsilon}\dot{\chi}) = f_\chi + m_{\mathsf{a}}' g \sin\phi - m_{\mathsf{a}}' b \dot{\Omega}_{\mathsf{b}} \cos\chi\,,$$

$$\left(m_{\mathsf{a}} + \frac{m_{\mathsf{a}}''^2}{m_{\mathsf{f}}}\right)(\ddot{\epsilon} - \epsilon\dot{\chi}^2) = f_\epsilon - m_{\mathsf{a}}' g \cos\phi - m_{\mathsf{a}}' b \dot{\Omega}_{\mathsf{b}} \sin\chi\,,$$

$$\tfrac{1}{2} m_{\mathsf{a}} a^2 \dot{\Omega}_{\mathsf{a}} = T_{\mathsf{a}} \qquad \kappa(t) \equiv \frac{\epsilon(t)}{\delta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2} m_{\mathsf{a}} a^2 \Omega_{\mathsf{a}} + (M + m_{\mathsf{b}}) b^2 \Omega_{\mathsf{b}} + \left(m_{\mathsf{a}} + \frac{m_{\mathsf{a}}''^2}{m_{\mathsf{f}}}\right)\epsilon^2\dot{\chi} + m_{\mathsf{a}}' b \frac{\mathrm{d}}{\mathrm{d}t}(\epsilon \sin\chi)\right]$$

$$+ m_{\mathsf{a}}' b \dot{\Omega}_{\mathsf{b}} \epsilon \cos\chi \approx M g b \sin\alpha + m_{\mathsf{a}}' g \epsilon \sin\phi$$

$$f_\epsilon = -\frac{12\nu a m_{\mathsf{a}}''}{\delta^2} \frac{\dot{\kappa}}{(1-\kappa^2)^{3/2}}\,, \qquad f_\chi = \frac{12\nu a m_{\mathsf{a}}''}{\delta^2} \frac{\kappa(\Omega_{\mathsf{a}} + \Omega_{\mathsf{b}} - 2\dot{\chi})}{(2+\kappa^2)\sqrt{1-\kappa^2}}$$

$$T_{\mathsf{a}} = \frac{12\nu a m_{\mathsf{a}}''}{\delta} \frac{(1-\kappa^2)(\Omega_{\mathsf{b}} - \dot{\chi}) - (1+2\kappa^2)(\Omega_{\mathsf{a}} - \dot{\chi})}{3(2+\kappa^2)\sqrt{1-\kappa^2}}$$

Solution of the reduced equations

☞ If the slope is not too large, we find rocking solutions in which the inner cylinder slowly sediments towards the outer cylinder.

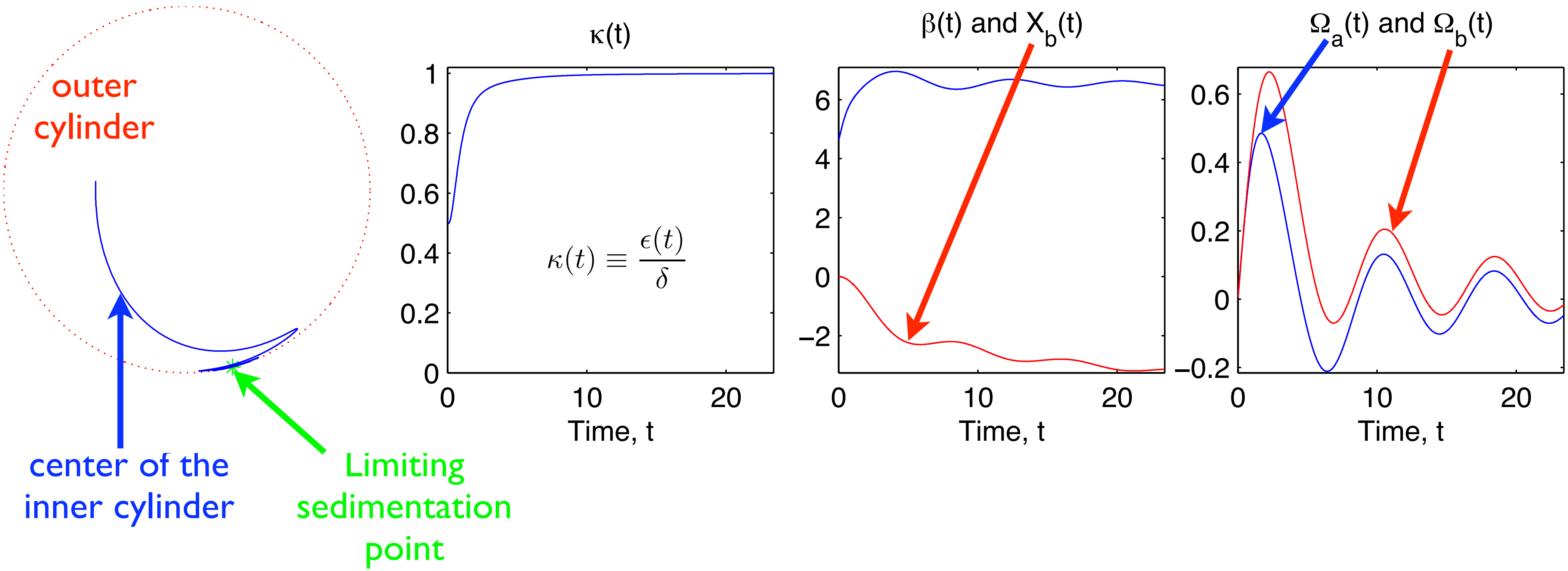
☞ If the slope is large, the system locks onto a runaway rolling solution with concentric cylinders:

$$\dot{X}_{\mathsf{b}} = \frac{M g \sin\alpha}{M + m_{\mathsf{b}} + \frac{1}{2} m_{\mathsf{a}}}\, t$$

☞ There is a range of slopes for which both rocking and rolling solutions co-exist (depending on ICs).

The decisive slope parameter is: $s \equiv \frac{a}{\delta} \frac{M}{{m_{\mathsf{a}}}'} \sin\alpha$

Some details of the rocking solutions:

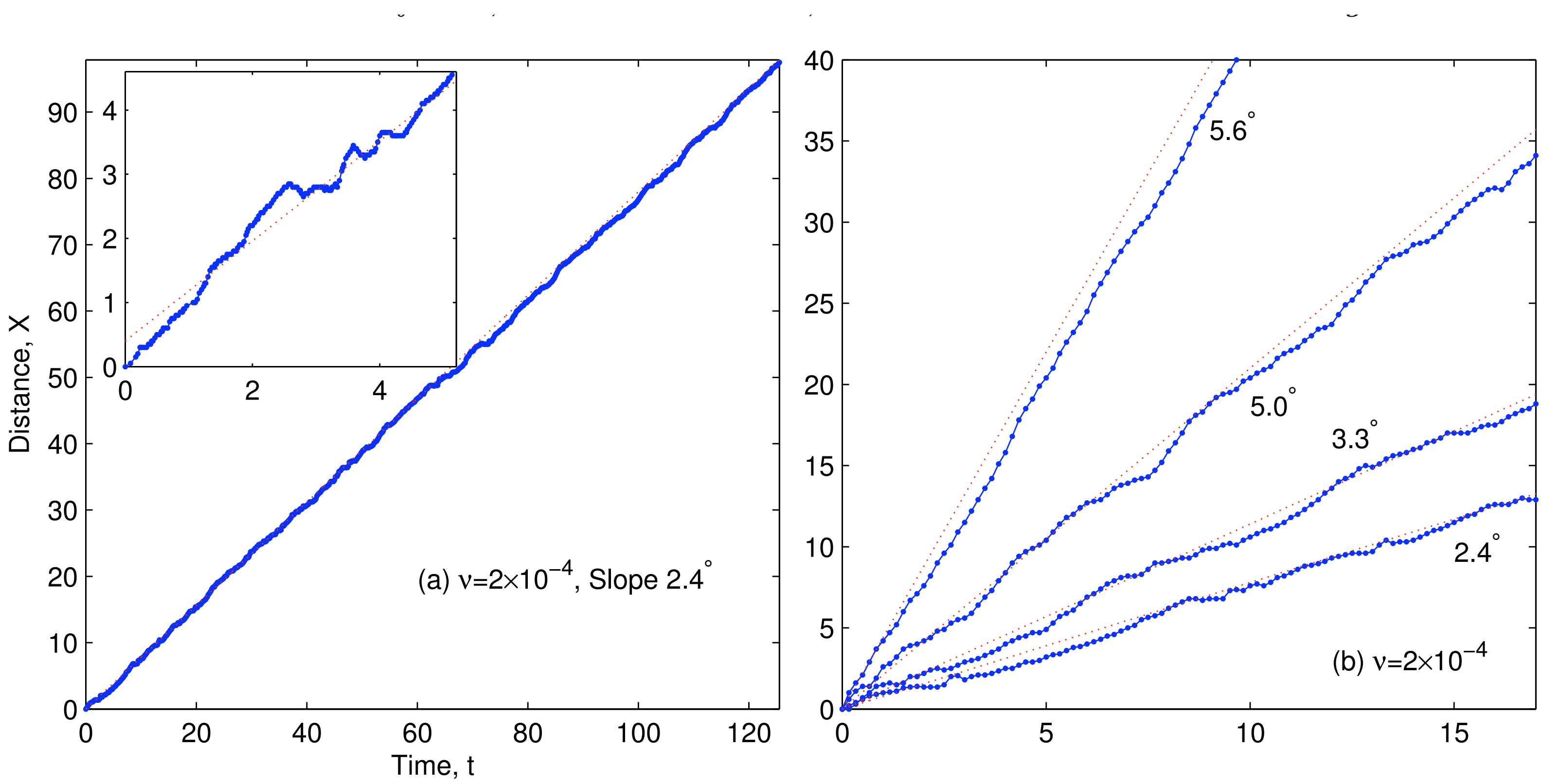


☞ The gap closes: $1 - \kappa(t) \propto t^{-1}$

☞ Power-law deceleration:

$$\Omega_{\mathrm{b}} \propto t^{-q}\,, \qquad X_{\mathrm{b}}(t) \propto t^{1-q}\,, \qquad q \equiv \frac{3(1+4\mu^2)}{2(1+2\mu)^2}\,, \qquad \mu \equiv \frac{M + m_{\mathrm{b}}}{m_{\mathrm{a}}}$$

Experiments with the snail-cylinder



☞ There is no indication of power-law deceleration.

☞ The experimental results are (far) simpler than the theoretical model.....

Hypothesis

Asperities prevent the closure of the gap and maintain an effective minimum separation:

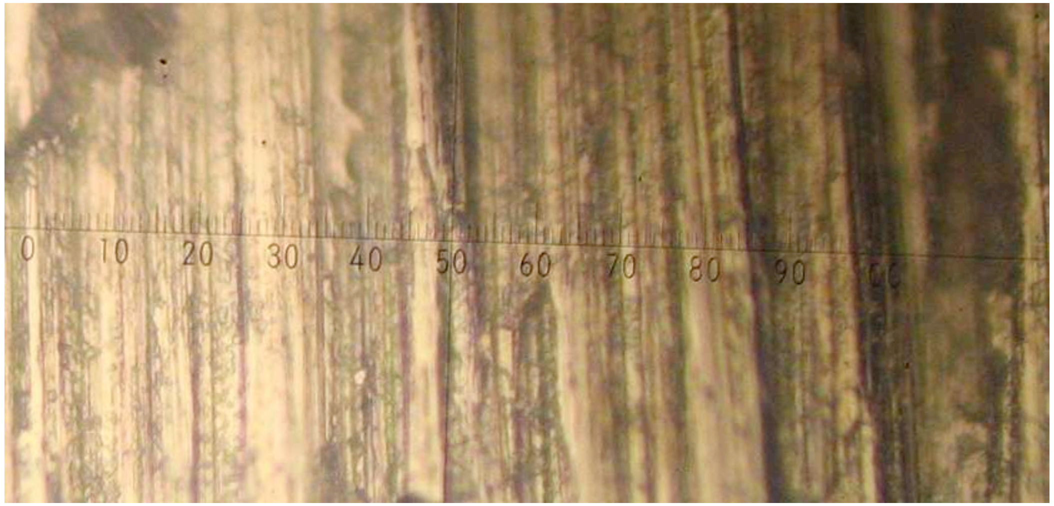
$$\kappa(t) \equiv \frac{\epsilon(t)}{\delta} \leqslant \kappa_* = \frac{\epsilon_*}{\delta}$$

We include a contact force with a “friction angle”:

$$|\mathcal{C}_\chi| \leqslant |\mathcal{C}_\epsilon| \tan \psi,$$

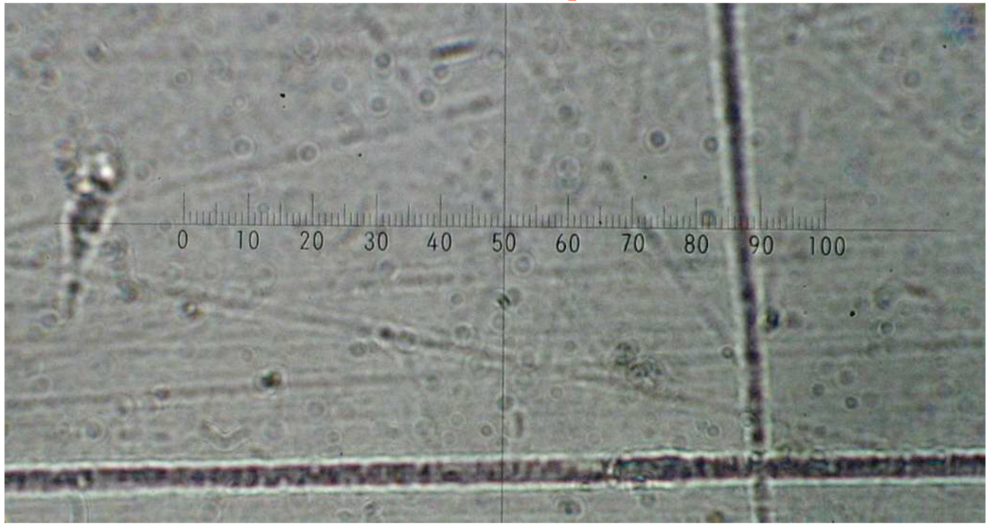
The radial equation and the torque equation must be modified once contact occurs.

Microscope images of the surface

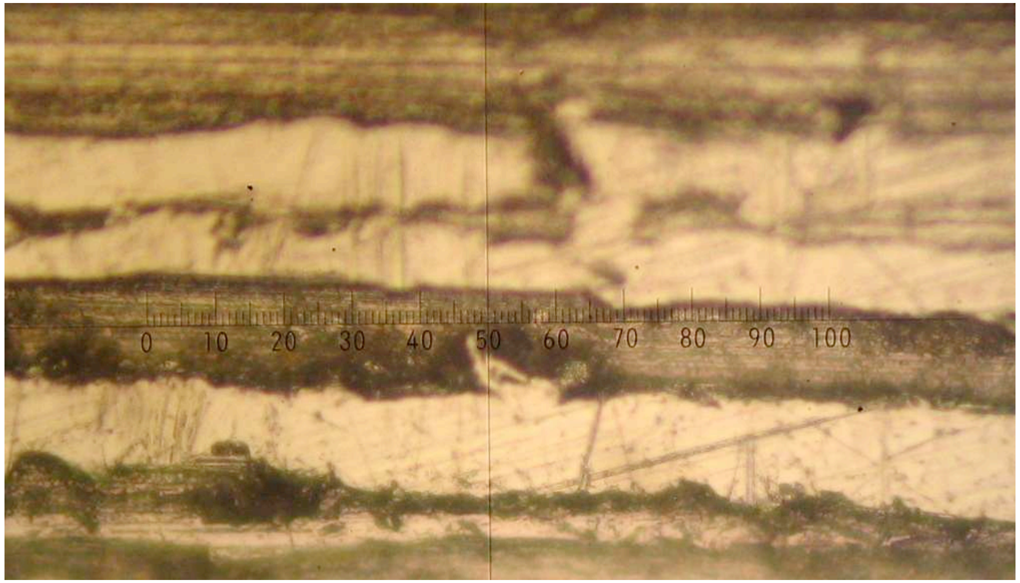


Steel

Perspex



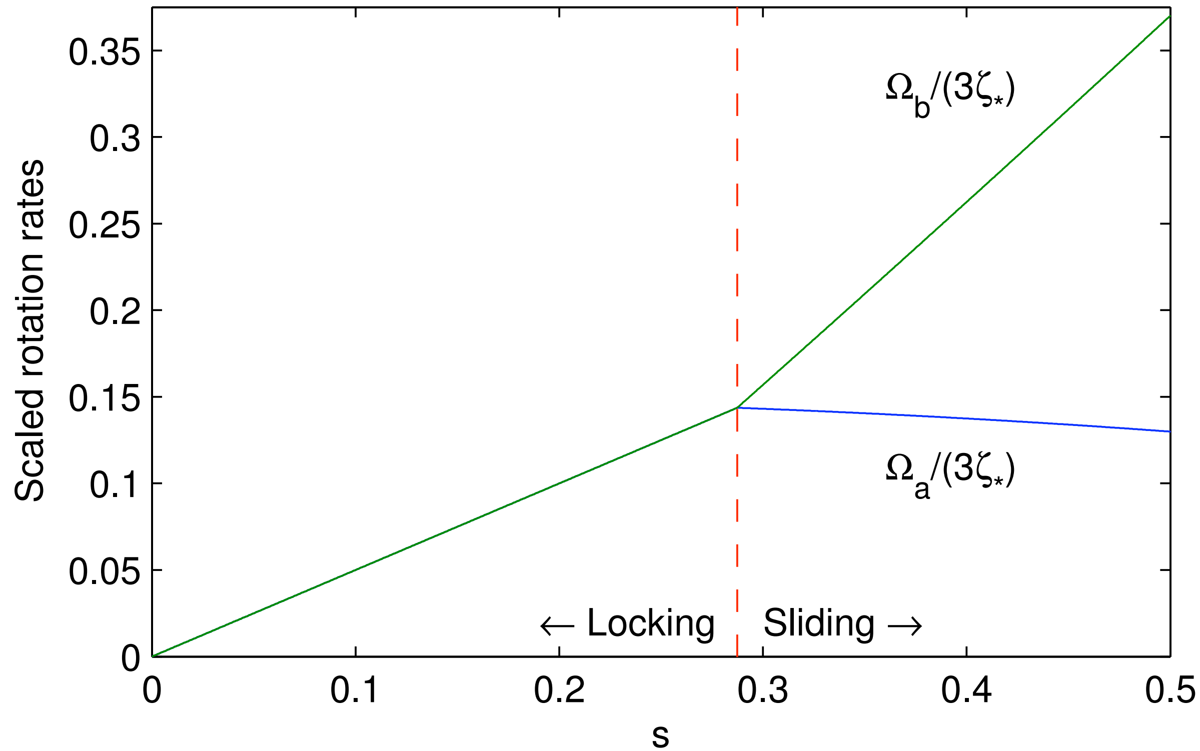
Aluminium



The scale is 250 microns in total length

Suggests the roughness scale is indeed of the order of tens of microns.

Predictions of the contact theory



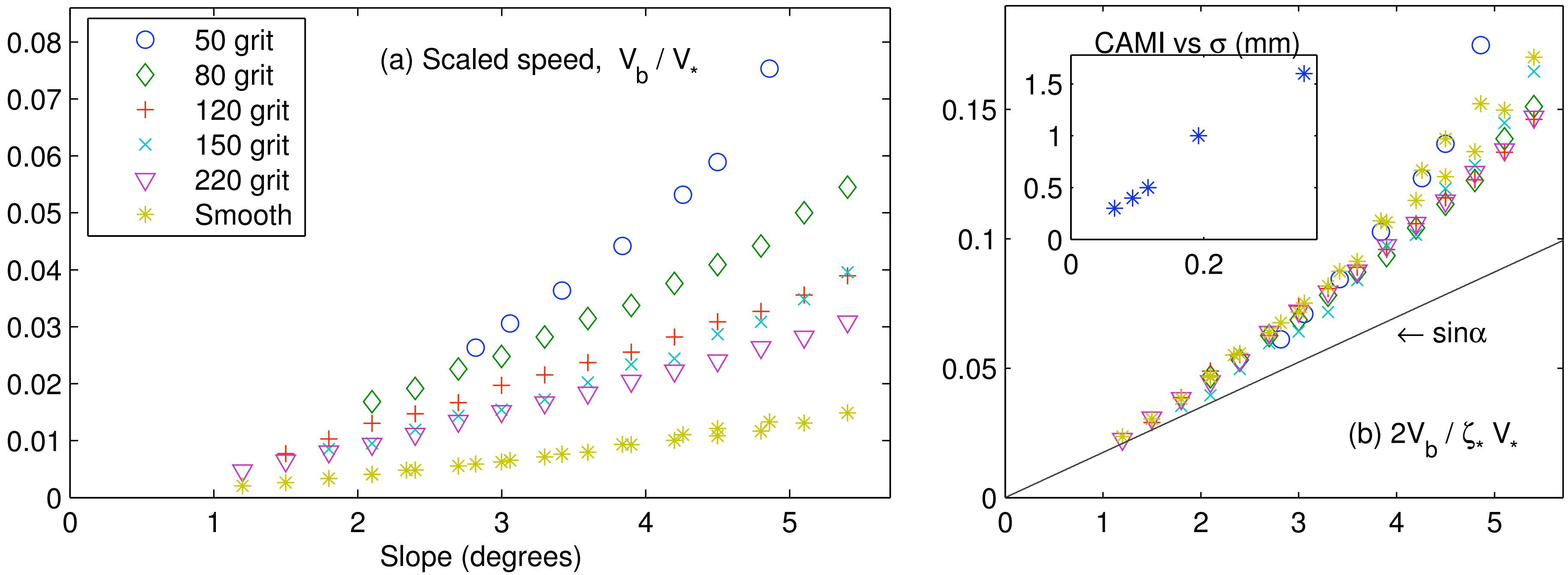
$$V = V_* \times \sin\alpha \times \zeta_* \,\mathrm{Max}\left(\tfrac{1}{2}, 1 - \frac{\sqrt{1-s^2}}{s}\tan\psi\right)$$

where

$$V_* = \frac{bM\delta g}{4\nu m''_{\mathrm{a}}},$$

$$\zeta_* \equiv \sqrt{1-\kappa_*^2} \ll 1,$$

Wrap the inner-cylinder with sandpaper



Grade	220	150	120	80	50	Smooth
Inferred (mm)	0.3	0.4	0.5	1	1.6	0.05
Expected (mm)	0.07	0.09	0.12	0.2	0.36	

Expected = average particle diameter quoted by the CAMI standard

For small slopes the contact force prevents slipping and the system rolls with constant speed

$$V = \underbrace{\frac{bMg\delta}{4\nu m_a''}}_{\equiv V_*} \times \sin\alpha \times \underbrace{\sqrt{1-\kappa_*^2}}_{\equiv \zeta_*}$$

Conclusions

☞ Sandpaper experiments show that surface roughness controls the speed of descent in rocking regime.

☞ The κ_✳ theory has some success in rationalizing experimental results. But κ_✳ is not completely convincing.

$$V = \underbrace{\frac{bMg\delta}{4\nu m_{\mathrm{a}}''}}_{\equiv V_*} \times \sin\alpha \times \underbrace{\sqrt{1-\kappa_*^2}}_{\equiv \zeta_*}$$

For example, the experimental dependence on α is not this simple.

☞ Nonetheless, I am now officially declaring victory over the snail cylinder and the snail ball.

☞ Other inclined-plane problems are diverting....

A Granular Snail Cylinder