SIO203C/MAE294C, Spring 2009, Final

8:00am to 11:00am, closed book

Problem 1

Consider the PDE:

$$\rho_t + \rho_x = \mathrm{e}^{-x}\left(1 - \int_0^\infty \rho(x,t)\mathrm{d}x\right), \quad \rho(x,0) = 0, \quad \rho(0,t) = 0,$$

in the domain $x > 0$ and $t > 0$. *(i)* Before attempting to solve the PDE, find

$$m(t) \equiv \int_0^\infty \rho(x,t)\,\mathrm{d}x,$$

and

$$\bar{x}(t) \equiv m^{-1}\int_0^\infty x\rho(x,t)\,\mathrm{d}x.$$

(ii) Solve the PDE, and sketch the solution as a function of x at $t = 1$.

Problem 2

Here are four PDE's

$$(1) \quad u_{1t} + u_1 u_{1x} = x, \qquad (2)\; u_{2t} + u_2 u_{2x} = \alpha u_2,$$
$$(3) \quad u_{3t} + u_3 u_{3x} = -\alpha u_3^2, \qquad (4) \quad u_{4t} + u_4 u_{4x} = 0.$$

In PDE's (2) and (3), α is a positive constant. Figure 1 shows four characteristic diagrams; the initial condition is

$$u_n(x,0) = \frac{1}{1+x^2}$$

in every case. Match the diagram with the PDE. Lucky guesses don't count, so explain your reasoning in thirty words or less.

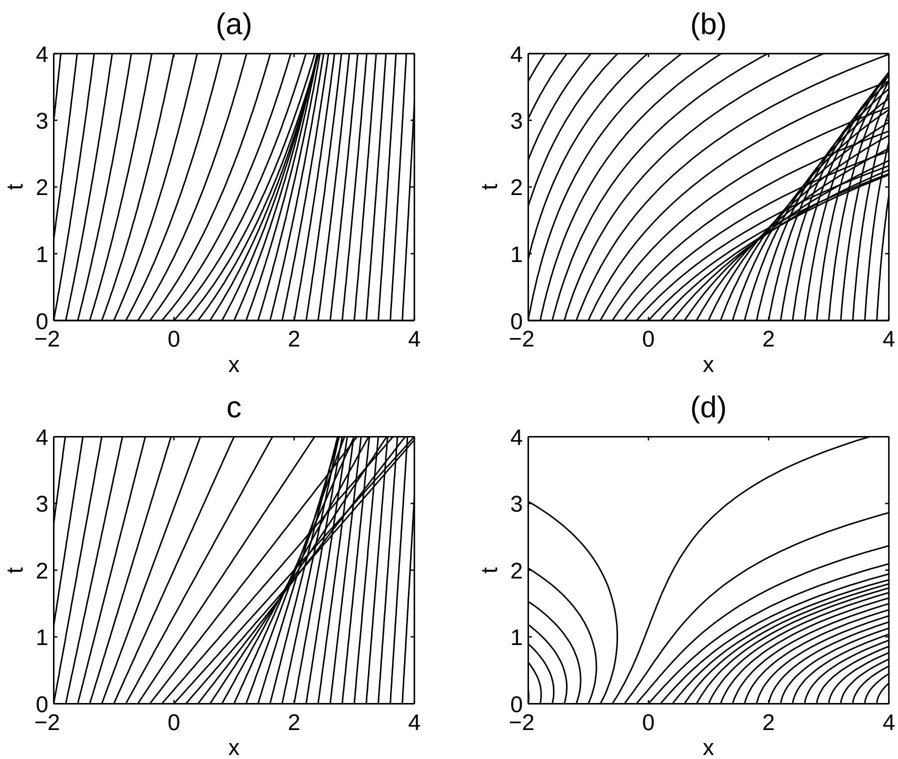

Figure 1: Match the characteristic diagram to the PDE's in **problem 2**.

Problem 3

If a is a positive real number then

$$\frac{1}{x^2+a^2} = \mathcal{F}^{-1}\left[\frac{\pi}{a}\mathrm{e}^{-a|k|}\right] = \frac{\pi}{a}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx-a|k|}\,\frac{\mathrm{d}k}{2\pi}\,.$$

$\mathcal{F}$ is the fourier transform and $\mathcal{F}^{-1}$ the inverse Fourier transform.

(i) Use the information above to evaluate the related inverse Fourier transforms

$$f_1(x) \equiv \mathcal{F}^{-1}\left[\mathrm{i}k \times \frac{\pi}{a}\mathrm{e}^{-a|k|}\right]\,,$$
$$f_2(x) \equiv \mathcal{F}^{-1}\left[|k| \times \frac{\pi}{a}\mathrm{e}^{-a|k|}\right]\,.$$

(ii) Consider Laplace's equation

$$u_{xx} + u_{yy} = 0\,,$$

in the half-plane $-\infty < x < \infty$ and $0 < y < \infty$, with the condition that $u \to 0$ as $|x| \to \infty$ and $y \to \infty$. There is a prescribed boundary value $u(x,0) = f(x)$. Express $u(x,y)$ in terms of the Fourier transform of the boundary function

$$\tilde{f}(k) \equiv \mathcal{F}[f; x \to k] = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x\,.$$

(An unevaluated inverse Fourier transform for $u(x,y)$ is fine — no convolutions required here.) *(iii)* Now suppose that the boundary condition is

$$f(x) = \frac{1}{x^2+a^2}\,, \quad \text{with the transform} \quad \tilde{f}(k) = \frac{\pi}{a}\mathrm{e}^{-a|k|}\,.$$

Find $u(x,y)$ by inverting the Fourier transform $\tilde{u}(k,y)$.

Problem 4

Consider the initial value problem

$$u_{tt} = u_{xx}\,, \quad u(x,0) = u_t(x,0) = 0\,, \quad u(0,t) = t\mathrm{e}^{-t}\,.$$

$\mathrm{e}^{-1} = 0.37$
$\mathrm{e}^{-5} \approx 0.0067$

The domain is $x > 0$ and $t > 0$. *(i)* Sketch the function $u(0,t) = t\exp(-t)$ on the interval $0 < t < 5$. Find the time t_* at which this function is a maximum, and indicate this on your sketch. *(ii)* Solve the PDE. Your solution should include an x-t diagram showing the separation between region in which $u(x,t) = 0$ and the region in which $u(x,t)$ is nonzero. *(iii)* Sketch u as a function of x at $t = 5$. *(iv)* Find the position, x_*, at which $u(x,5)$ achieves its maximum value and indicate this position on your sketch of $u(x,5)$.

Problem 5

The PDE

$$\psi_{tt} + \psi_{xxxx} = 0$$

has an energy conservation law of the form

$$\mathcal{E}_t + \mathcal{J}_x = 0\,,$$

where the energy density is $\mathcal{E} = \frac{1}{2}\psi_t^2 + \cdots$ and the energy flux is $\mathcal{J} = \cdots$. Find complete expression for $\mathcal{E}$ and $\mathcal{J}$ (i.e., fill in the $\cdots$).