SIO203C/MAE294C, Spring 2010, Final

8:00am to 11:00am

Problem 1

Solve the PDE $u_t + xu_x = 0$ with the initial condition $u(x, 0) = \cos x$.

Problem 2

Consider the PDE

$$u_t = (\beta|x|u_x)_x \,, \qquad u(x, 0) = \delta(x)\,.$$

(i) What are the dimensions of β and the dimensions of $u(x, t)$? *(ii)* Determine

$$m(t) \equiv \int_{-\infty}^{\infty} u(x, t)\, \mathrm{d}x\,.$$

(iii) The PDE might have a similarity solution of the form

$$u(x, t) = t^a U\left(\frac{|x|}{\beta t^b}\right)\,. \tag{1}$$

Find the exponents a and b that are consistent with the PDE *and* with the initial condition. *(iv)* Find this similarity solution.

Problem 3

(i) Starting with

$\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ the inverse Fourier transform.

$$\mathrm{e}^{-\frac{1}{2}k^2} = \mathcal{F}\left[\frac{\mathrm{e}^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}; x \to k\right] = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x\,, \tag{2}$$

evaluate the related Fourier transforms

$$\mathcal{F}\left[x\frac{\mathrm{e}^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\right]\,, \qquad \text{and} \qquad \mathcal{F}\left[\frac{\mathrm{e}^{-\frac{1}{2}x^2 - x}}{\sqrt{2\pi}}\right]\,. \tag{3}$$

(ii) Consider Laplace's equation

$$u_{xx} + u_{yy} = 0\,,$$

in the upper half-plane $0 < y < \infty$, with the condition that $u(x, y) \to 0$ as $|x| \to \infty$ and $y \to \infty$. There is a prescribed boundary value $u(x, 0) = f(x)$. Express the normal derivative at the boundary, that is $u_y(x, 0)$, in terms of the Fourier transform of the boundary function

$$\tilde{f}(k) \equiv \mathcal{F}\,[f; x \to k] = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x\,.$$

(An unevaluated inverse Fourier transform for $u_y(x, 0)$ is fine — no convolutions required here.) *(iii)* Now suppose that the boundary function is

$$u(x, 0) = \underbrace{\frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}}}_{=f(x)}\,. \tag{4}$$

Find $u_y(0, 0)$. Hint: the answer is not zero. If you get zero, check your answer to part *(ii)*.

Problem 4

Evaluate the two-dimensional line integral

$$J = \int_{\mathcal{C}} \ln r\,\mathrm{d}\ell\,, \tag{5}$$

$r = \sqrt{x^2 + y^2}$

where the path of integration $\mathcal{C}$ is a circle with radius one centered on the point $(x, y) = (3, 4)$ i.e., $\mathcal{C}$ is the curve $(x-3)^2 + (x-4)^2 = 1$.

Problem 5

Suppose that $u(x, t)$ is a conserved density satisfying the conservation equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0\,, \qquad \text{with initial condition} \qquad u(x, 0) = -x\mathrm{e}^{-x^2}\,. \tag{6}$$

At what time and location does the shock first form? Find the location and strength of the developed shock.

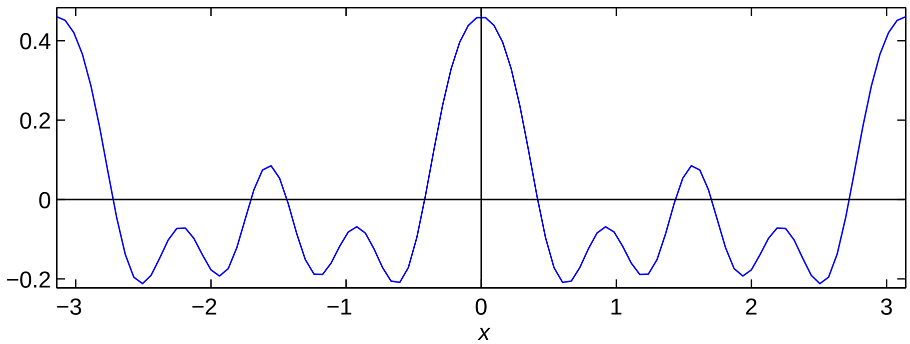

Figure 1: Figure for **problem 6**.

Problem 6

Figure 1 shows a function $f(x)$ defined on the interval $-\pi < x < \pi$. The function can be represented by a Fourier series

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \,. \tag{7}$$

Identify the non-zero coefficients in (7). Explain your reasoning in twenty or thirty words.

Problem 7

(i) Find the general solution of the PDE

$$\cosh x \, u_{tt} - \left(\frac{u_x}{\cosh x}\right)_x = 0 \,, \tag{8}$$

in terms of two arbitrary functions. *(ii)* Solve the PDE with the initial condition

$$u(x,0) = \frac{1}{\cosh x} \,, \qquad u_t(x,0) = 0 \,. \tag{9}$$

To check your answer, show that $u(0,t) = 1/\sqrt{1+t^2}$. *(iii)* Show that the PDE in (8) has an energy conservation law,

$$E_t + J_x = 0 \,, \tag{10}$$

and find expressions for the energy density E and flux J in terms of u_t, u_x, $\cosh x$ etc.