SIO203C/MAE294C, Spring 2018, Final

3:00pm to 6:00pm

Problem 1

Find the general solution of $yu_x + xu_y = y$ as the sum of a particular solution and an arbitrary function.

Problem 2

Paint flowing down a wall has a thickness $\eta(x,t)$ governed by the conservation law

$$\eta_t + \left(\tfrac{1}{3}\eta^3\right)_x = 0\,. \tag{1}$$

A stripe of paint is applied at $t = 0$ with initial thickness

$$\eta(x,0) = \begin{cases} 1\,, & 0 < x < 1\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{2}$$

Find $\eta(x,t)$ and sketch η as a function of x at $t = 1$ and at $t = 2$.

Note:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \sqrt{2\pi}\,\mathrm{e}^{-k^2/2}$$

and

$$\int_{-\infty}^{\infty} \mathrm{e}^{-|x|} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{2}{1+k^2}$$

Problem 3

Solve the integral equation

$$\mathrm{e}^{-x^2/2} = \int_{-\infty}^{\infty} \mathrm{e}^{-|x-v|} f(v)\,\mathrm{d}v\,. \tag{3}$$

Check your answer by showing that $f(0) = 1$.

Problem 4

(i) Use the Fourier transform to obtain an integral representation of the solution $g(x,t)$ of the hyperdiffusion equation

$$g_t + \nu g_{xxxx} = 0\,, \qquad \text{with IC} \quad g(x,0) = \delta(x)\,. \tag{4}$$

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^{\infty} \mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t$$

(ii) Show that

$$g(0,t) = \frac{\Gamma(c)}{\pi(\nu t)^{1/4}} \tag{5}$$

$$z\Gamma(z) = \Gamma(z+1)$$

where c is a constant you should calculate.

Problem 5

Note:

$\operatorname{sech} x = \dfrac{1}{\cosh x}$

(i) Find the general solution of the PDE

$$\cosh x \, U_{tt} - (\operatorname{sech} x \, U_x)_x = 0 \,, \tag{6}$$

in terms of two arbitrary functions. *(ii)* Solve the PDE with the initial condition

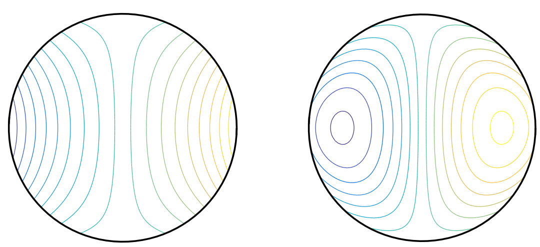
$$U(x,0) = \operatorname{sech} x \,, \qquad U_t(x,0) = 0 \,. \tag{7}$$

To check your answer, show that $U(0,t) = 1/\sqrt{1+t^2}$. *(iii)* Show that the PDE in (6) has an energy conservation law,

$$E_t + J_x = 0 \tag{8}$$

and find expressions for the energy density E and flux J in terms of U_t, U_x, $\cosh x$ etc.

Problem 6



(i) Solve the $d = 2$ Laplace equation $\phi_{xx} + \phi_{yy} = 0$ inside the unit disc (i.e., $0 < r < 1$) with the boundary condition $\phi = x^3$ on $r = 1$. *(ii)* The figure shows the contours of two functions within the unit disc. Which panel (left or right) shows the harmonic function from part *(i)*? Hint: Thirty words or less and you can answer part *(ii)* without having done part *(i)*.

$r = \sqrt{x^2 + y^2}$