SIO203CMAE294C, Spring 2009, Mid-term

8:05am to 9:45am, closed book

Turn the page!

Problem 1

Find the function $u(x,y)$ defined by the PDE

$$u_x = 0\,,$$

with the condition that $u(x,y) = x$ on the curve $y = \ln x$.

Problem 2

Find the solution $u(x,y,z)$ of

$$u_x + u_y + u_z = 0\,, \qquad \text{and} \qquad u_x - u_y + 2u_z = 0\,,$$

in terms of an arbitrary function of one variable.

Problem 3

(i) Consider the PDE

$$u_x + 2xu_y = 1\,,$$

with $u(-\infty < x < \infty, 0) = x$. Solve the PDE, and sketch the domain of definition of the solution in the (x,y)-plane.

Problem 4

Consider the PDE

$$u_t + uu_x = 0\,, \qquad u(x,0) = \frac{1}{1+\mathrm{e}^x}\,.$$

(i) Using sketches of the characteristic diagram and the "sliding construction", indicate how the solution evolves towards a shock. *(ii)* Find the time t_s at which $u(x,t)$ first becomes multivalued.

Problem 5

Consider the PDE:

$$\rho_t + \rho_x = \mathrm{e}^{-x}\left(1 - \int_0^\infty \rho(x,t)\mathrm{d}x\right), \qquad \rho(x,0) = 0\,, \qquad \rho(0,t) = 0\,,$$

in the domain $x > 0$ and $t > 0$. *(i)* Before attempting to solve the PDE, find

$$m(t) \equiv \int_0^\infty \rho(x,t)\,\mathrm{d}x\,.$$

(ii) Solve the PDE, and sketch the solution as a function of x at $t = 1$.

Problem 6

Consider the PDE

$$u_t = \nu\left(x^{1/2}u_x\right)_x\,, \qquad (\spadesuit)$$

which is posed in the domain $x \geq 0$ and $t \geq 0$, with the initial and boundary conditions

$$u(x,0) = 0\,, \qquad \text{and} \qquad u(0,t) = 1\,.$$

(i) What are the dimensions of ν? *(ii)* In thirty words or less, give a dimensional argument indicating that there is a similarity solution of the form

$$u = U(\xi)\,, \qquad \text{with} \qquad \xi \equiv \frac{x}{(\nu t)^?}\,. \qquad (\clubsuit)$$

Your argument should determine the exponent "?" in ($\clubsuit$). *(iii)* Substitute the similarity solution in ($\clubsuit$) into ($\spadesuit$), and solve the resulting ordinary differential equation for $U(\xi)$ in terms of the function

$$F(z,\alpha,\beta) \equiv \int_z^\infty \mathrm{e}^{-\alpha q^\beta}\,\frac{\mathrm{d}q}{\sqrt{q}}\,.$$

Make sure your solution satisfies the initial and boundary conditions. *(iv)* Consider the related problem

$$v_t = \nu\left(x^{1/2}v_x\right)_x\,,$$

with the initial and boundary conditions

$$v(x,0) = 0\,, \qquad \text{and} \qquad v(0,t) = \begin{cases} 1\,, & \text{if } 0 < t < 1, \\ 0\,, & \text{if } 1 < t\,. \end{cases}$$

Express $v(x,t)$ in terms of $u(x,t)$.