SIO203CMAE294C, Spring 2019, Mid-term

3:30pm to $\sim$ 4:50pm, open notes

Problem 1

(i) Find the general solution of the PDE $xu_x + yu_y = 0$ in terms of an arbitrary function. *(ii)* Find the solution $u(x,y)$ that satisfies the data $u(x,x^2) = \mathrm{e}^x$.

Problem 2

Find the time at which the PDE $u_t + uu_x = 0$, with the initial condition $u(x,0) = -x/(1+x^2)$, first becomes multi-valued i.e., the *shock-time*, t_s. Assuming that $u(x,t)$ is a conserved density, find the location and the strength of the shock when $t > t_s$.

Problem 3

In class we solved Stokes's second problem: the diffusion equation $u_t = \kappa u_{xx}$, on the half-line $(x > 0)$, with boundary and initial conditions

$$u(0,t) = 1\,, \qquad \text{and} \qquad u(x,0) = 0\,. \tag{1}$$

In terms of the similarity variable, $\eta = x/2\sqrt{\kappa t}$, here is the solution:

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_\eta^\infty \mathrm{e}^{-\alpha^2} \mathrm{d}\alpha = \mathrm{erfc}(\eta)\,. \tag{2}$$

(i) Using the result above, solve the half-line $(x > 0)$ problem $v_t = \kappa v_{xx}$, with boundary and initial conditions

Hint: erf+erfc = 1

$$v(0,t) = 0\,, \qquad \text{and} \qquad v(x,0) = 1\,. \tag{3}$$

(ii) Now solve the half-line $(x > 0)$ problem

$$w_t = \kappa w_{xx} + 1\,, \tag{4}$$

with boundary and initial conditions

$$w(0,t) = 0\,, \qquad w(x,0) = 0\,. \tag{5}$$

(iii) Show that

$$\kappa w_x(0,t) = \beta\sqrt{\kappa t}\,, \tag{6}$$

and find the value of the constant β.

Problem 4

In recitation, and in the notes, we obtained the Fourier series

$$\mathrm{e}^{-\alpha x} = \frac{\sinh \alpha\pi}{\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m \mathrm{e}^{\mathrm{i}mx}}{\alpha + \mathrm{i}m}, \qquad \text{for} \qquad -\pi < x < \pi\,. \tag{7}$$

(i) Show that

$$\pi \coth \alpha\pi = \sum_{m=-\infty}^{\infty} \frac{\alpha}{\alpha^2 + m^2}\,. \tag{8}$$

(ii) On the interval $-\pi < x < \pi$, the Fourier series representation of $(x^2 - \pi^2)\mathrm{e}^{\alpha x}$ might be

$$(x^2 - \pi^2)\mathrm{e}^{-\alpha x} \stackrel{?}{=} 2 \sum_{m=-\infty}^{\infty} \frac{(-1)^m \mathrm{e}^{\mathrm{i}mx}}{(\alpha + \mathrm{i}m)^2} \left[\frac{\sinh \pi\alpha}{\pi} - \frac{\cosh \pi\alpha}{\alpha + \mathrm{i}m} \right]. \tag{9}$$

Or perhaps it should be

$$(x^2 - \pi^2)\mathrm{e}^{-\alpha x} \stackrel{?}{=} 2 \sum_{m=-\infty}^{\infty} \frac{(-1)^m \mathrm{e}^{\mathrm{i}mx}}{\alpha + \mathrm{i}m} \left[\frac{\sinh \pi\alpha}{\pi} - \frac{\cosh \pi\alpha}{\alpha + \mathrm{i}m} \right]? \tag{10}$$

Which expression, (9) or (10), might be correct? Lucky guesses don't count, so explain your reasoning in ten or twenty words. (Little or no algebra here.)

Problem 5

Consider an age-stratified population, with histogram $h(a,t)$ satisfying

$$h_t + h_a = -\mathrm{e}^{-t} h\,. \tag{11}$$

The initial condition is

$$h(a,0) = N\mathrm{e}^{-a}\,, \tag{12}$$

and the birth rate $h(0,t)$ is adjusted so that the population is constant:

$$\forall t \geq 0: \qquad N = \int_0^{\infty} h(a,t)\,\mathrm{d}a\,. \tag{13}$$

Solve the PDE and exhibit $h(a,t)$.