

## Homogenization of potential vorticity in planetary gyres

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The mean circulation of planetary fluids tends to develop uniform potential vorticity  $q$  in regions where closed time-mean streamlines or closed isolines of mean potential vorticity exist. This state is established in statistically steady flows by geostrophic turbulence or by wave-induced potential-vorticity flux. At the outer edge of the closed contours the expelled gradients of  $q$  are concentrated. Beyond this transition lies motionless fluid, or a different flow regime in which the planetary gradient of  $q$  may be dominant. The homogenized regions occur where direct forcing by external stress or heating within the closed isoline is negligible, upon the potential-density surface under consideration. In the stably stratified ocean such regions are found at depths greater than those of direct wind-induced stress or penetrative cooling. In 'channel' models of the atmosphere we again find constant  $q$  when mesoscale eddies cause the dominant potential-vorticity flux. In the real atmosphere the results here can apply only where internal heating is negligible. The derivations given here build upon the Prandtl-Batchelor theorem, which applies to non-rotating, steady two-dimensional flow. Supporting evidence is found in numerical circulation models and oceanic observations.

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### 1. Introduction

In the literature of fluid mechanics there are several studies of steady two-dimensional closed-streamline flow, in which weak molecular diffusion causes the gradient of some conservative scalar property to be expelled from a region enclosed by streamlines. In the first volume of this journal Batchelor (1956) established the result (earlier noted by Prandtl) that the relative vorticity thus becomes uniform in two-dimensional steady closed gyres of fluid without planetary rotation. Weiss (1966) and Proctor (1975) (for a review see Moffatt 1978), found that the gradient of the magnetic potential is expelled from two-dimensional steady thermal convection cells, and this production of concentrated, energetic 'flux ropes' appears in far more complex flows, as in the solar photosphere. Ingersoll (1969) showed that the core of a two-dimensional Taylor column, surrounded by a closed streamline, must be stagnant. Benney & Bergeron (1969), Davis (1969) and Redekopp (1980) similarly determined the closed circulation in the 'cat's-eye' of the nonlinear critical layer associated with various kinds of waves.

In the study of large-scale atmospheric and oceanic circulation the principal field

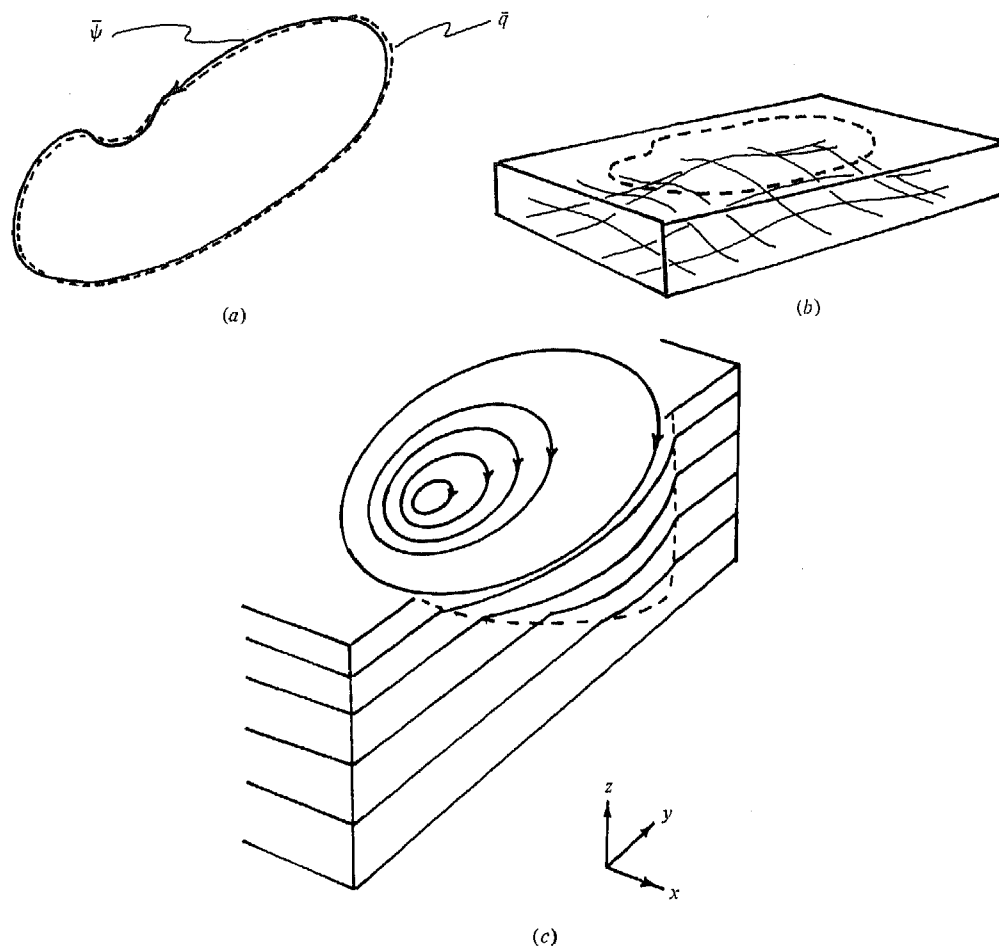


FIGURE 1. (a) Sketch of the circulation path  $\psi = \text{const}$  and the geostrophic contour of  $\bar{q} = \text{const}$ . Both curves lie on a surface of constant potential density (which lies near a level of geopotential surface). (b) The circulation gyres under consideration may occur in single layers of homogeneous fluid, or (c) in the quasi-horizontal flow in a stratified fluid. For large-scale flows with small relative vorticity, the potential vorticity is simply  $\bar{q} \approx f/h$ ; uniform  $\bar{q}$  could occur if isopycnal layers thicken to the north.

variable, the potential vorticity  $q$ , is quasi-conservative.† In this paper we establish that  $q$  may indeed be expected to become uniform within closed streamlines. In the oceanic case this leads to a theory of the general circulation.

The simplest illustration of expulsion would involve a purely passive scalar field, like the concentration of a neutrally buoyant dye. Here and in Batchelor's study the expelled quantity is active: a knowledge of  $q$  plus boundary conditions gives knowledge of the entire flow field.

†  $q$  is defined here to be  $H(f + \zeta)/h$ , where  $\zeta$  is the vertical relative vorticity,  $h$  the (variable, unsteady) thickness of a layer between constant density surfaces, and  $H$  the areal mean of  $h$ . By quasi-conservative we mean that  $q$  obeys  $Dq/Dt = G - \Delta$ , where  $D/Dt$  is the convective derivative, and  $G$  (forcing) and  $\Delta$  (dissipation) are both small relative to the individual terms in  $Dq/Dt$ .

The present problem is distinct from many of those cited above in being three-dimensional and in having, as a component of  $q$ , the large-scale north-south variation known as the  $\beta$ -effect.  $\beta$ , the northward gradient of the Coriolis frequency  $f$ , dominates the dynamics where the flow is sufficiently weak, and in much of the classical theory of ocean circulation and large-scale wave motion this gradient is supposed to remain dominant. We thus anticipate that there is a threshold of energy which must be surpassed if the flow is to produce closed  $q$ -contours.

Planetary fluids tend to be stably stratified in potential density  $\rho$ , to have mean rotation, and to be confined to a spherical shell by solid or virtually solid boundaries (figure 1). Often the heating or stresses that drive the fluid vanish over large volumes. For example the oceans are driven largely at or near their upper boundary. The inner  $n-2$  layers of an  $n$ -layer quasi-geostrophic ocean model are remote from direct wind forcing and strong frictionless effects, and obey

$$\bar{\mathbf{u}} \cdot \nabla \bar{q} = -\nabla \cdot \overline{\mathbf{u}'q'} + \left\{ \begin{array}{l} \text{terms due to the averaged effect of small-scale} \\ \text{processes such as internal waves} \end{array} \right\} \quad (1.1)$$

in the interior. (The overbar in (1.1) denotes a time-average; primes denote fluctuations about that average.) In regions where the eddy flux divergence  $\nabla \cdot \overline{q'\mathbf{u}'}$  is weak the right-hand side of (1.1) is small and a first integral is

$$\bar{q} = Q(\bar{\psi}, \bar{\rho}) + \{\text{corrections due eddy fluxes etc.}\}. \quad (1.2)$$

Because the fluid is confined to nearly horizontal density surfaces, the statement in the previous paragraph that planetary flows are three-dimensional is potentially misleading. In the non-diffusive limit  $q$  is conserved on fluid particles, and hence the three-dimensional problem retains some aspects of its two-dimensional predecessors. This reduced dimensionality has important dynamic consequences and simplifies many of the theoretical arguments in this article. For example, because particles are confined to density surfaces, the eddy flux of potential vorticity in (1.1) cannot transfer mean potential vorticity across density surfaces. Thus under restrictions discussed below  $\overline{\mathbf{u}'q'}$  can be expressed as a downgradient flux of mean potential vorticity (i.e.  $\overline{\mathbf{u}'q'} \cdot \nabla \bar{q} < 0$ ) within, but not across, mean-density surfaces (i.e.  $\overline{\mathbf{u}'q'} \cdot \nabla \bar{\rho} = 0$ ). In view of the difficulty of generalizing the Prandtl-Batchelor theorem, and other related arguments, to fully three-dimensional flows (e.g. Grimshaw 1969) the inability of geostrophic eddies to flux  $q$  into a region of closed streamlines from 'above' and 'below' is a welcome simplification.

However, one should not form the impression that nothing of importance can be transported across density surfaces by eddies. Rhines & Holland (1979) show that the eddy transport of potential vorticity appears as a force-like quantity in the averaged momentum equation. The corresponding stresses can transport momentum both laterally (i.e. within density surfaces), through Reynolds stresses, and vertically (i.e. across density surfaces), through form drag. In fact Rhines & Holland argue that for large-scale mean flows (i.e. flows in which the relative vorticity is negligible compared with the planetary vorticity and vortex stretching) the latter, *vertical* process dominates. Thus in the planetary-scale problems which concern us here, a lateral flux of potential vorticity is equivalent to a vertical flux of horizontal momentum.

The surfaces of constant potential density  $\bar{\rho}$  are covered with curves of constant  $\bar{q}$ , known as geostrophic contours or isostrophes, which coincide with streamlines. To

make further progress  $Q$  must be determined. If a geostrophic contour intersects a lateral boundary,  $Q$  is fixed by a zero-velocity boundary condition, except in a region where a viscous or inertial boundary layer can occur. This leads one to conclude that there is no motion at this order in interior subsurface layers whose  $\bar{q}$ -contours are connected to both eastern and western boundaries. (For further discussion see Rooth, Stommel & Veronis 1978; Rhines & Holland 1979.) If one applies linearized non-diffusive inviscid theory to the spin-up of the circulation from a state of rest (Anderson & Gill 1975) the dominance of  $\beta y/h$  over other contributors to the  $\bar{q}$ -field thus ensures that the circulation will eventually vanish on interior layers, for the contours of  $\beta y$  intersect coastal boundaries in both the east and west.

This trivial solution,  $\bar{v} = 0$ , can be avoided if the geostrophic contours and streamlines close upon themselves in the interior of the basin; however, in this case  $Q$  cannot be determined by lateral boundary conditions, and is in fact arbitrary within the context of an eddyless inviscid theory. It is in these regions, isolated from lateral boundaries, that we expect strong interior flows to develop.

This expectation is reinforced when one considers mean-flow induction round closed geostrophic contours from the perspective of the averaged momentum equations. As Welander (1969) emphasized, flow driven round 'free' (i.e. closed) contours is far more intense than flow in the presence of contours blocked by a rigid boundary. In order that the interior geostrophic contours close, however, the forcing in the upper ocean must deform the density interfaces sufficiently to overcome the  $\beta$ -effect in the layers below.

## 2. Formulation; separation of wind-driven and thermal ocean-circulations

The full ocean-circulation problem encompasses the development of the density field from surface heating, cooling, evaporation and precipitation. The equations describe the simultaneous evolution of the density and flow fields. Important to this complete problem is the intersection of constant-density surfaces with the upper boundary. Free paths are thus created, along which the external atmospheric forcing can communicate deep into the interior. The disparate time scales of changes in gyre-scale velocity and density fields (decades and centuries respectively) suggest, however, that it is meaningful to consider the development of wind-driven circulation on an essentially prescribed basic thermocline stratification. This separation is automatically achieved by adopting the quasi-geostrophic equations in which the unperturbed density is a function of the vertical co-ordinate  $z$ , only. Implicit in this ideal is the condition that fluid particles in the wind-driven circulation should circulate about a gyre several or many times before escaping from it. At some time in their history they pass through the upper boundary layer, where their density and potential vorticity are rapidly changed by interaction with the atmosphere, or eddy dispersion may transport them laterally to another gyre or arm of the circulation. But their residence time in the recirculating interior of a single gyre is long enough for lateral eddy  $q$ -transport gradually to alter their potential vorticity and so to establish a mean flow whose structure reflects the mean-field effects of eddies, rather than the source distribution of the upper mixed layer.

The simplicity of classical wind-driven ocean circulation theory is related to the apparent independence of the vertically integrated velocity from the density stratifi-

cation and from the vertical structure of the velocity. If the driving wind-stress is  $\tau$ , the linearized vorticity equation (1.1)

$$\beta \bar{v} = f \partial \bar{w} / \partial z$$

gives

$$\beta \int_{-d}^0 \bar{v} dz = \rho_0^{-1} \nabla \times \tau|_{z=0} - f \bar{w}(-d),$$

where  $(u, v, w)$  is the velocity vector, and  $(x, y, z)$  are (eastward, northward, upward) Cartesian  $\beta$ -plane co-ordinates. The interior is driven by vertical velocity  $w_0 = \nabla \times \tau / \rho_0 f$  at the base of the upper Ekman boundary layer. Under the assumption that  $\bar{w}$  vanishes at some depth  $d$ , the net transport  $\int \bar{v} dz$  is a known function of wind-stress. Observations (Leetma, Niiler & Stommel 1977) and numerical models (e.g. Holland & Rhines 1980) indeed suggest that with sufficient time- and space-averaging this 'Sverdrup balance' roughly holds in open ocean regions, far from western boundary currents. But theory is needed to calculate the vertical structure of  $\bar{\mathbf{u}}$ , and to justify the existence of a depth of vanishing  $\bar{w}$ . That theory is the principal application of the ideas given here.

*Equations.* We use the quasi-geostrophic equations familiar in synoptic meteorology and 'mesoscale' oceanography, in which rotation and buoyancy are significant,

$$\frac{Dq}{Dt} = \text{forcing} + \text{dissipation}, \tag{2.1}$$

where, in terms of the geostrophic stream function,

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \frac{\partial(\psi, \ )}{\partial(x, y)}.$$

Here  $\mathbf{u} = \hat{\mathbf{z}} \times \nabla \psi$ ,  $\hat{\mathbf{z}}$  being a vertical unit vector and  $\psi$  is proportional to the pressure. The  $\beta$ -plane approximation is taken. In terms of  $\psi$  one has

$$\left. \begin{aligned} q &= \nabla_{\frac{1}{2}}^2 \psi + \frac{\partial}{\partial z} \left( F \frac{\partial \psi}{\partial z} \right) + \beta(y - y_0), \\ F &= f_0^2 / N^2, \quad f = f_0 + \beta(y - y_0), \end{aligned} \right\} \tag{2.2}$$

where  $\nabla_{\frac{1}{2}}^2$  is the horizontal Laplacian and  $N$  the buoyancy frequency. This expression for  $q$  is equivalent to that given in the footnote above.

We will be primarily concerned with the interaction between the mean flow, characterized by velocity and horizontal length scales  $U$  and  $L$ , and mesoscale eddies, characterized by  $U'$  and  $L'$ . These scales are used to define non-dimensional variables, denoted temporarily by  $*$ ,

$$\bar{q} = \beta L \bar{q}_*, \quad \bar{\psi} = \bar{U} L \bar{\psi}_*, \quad q' = \beta L' q'_*, \quad \psi' = U' L' \psi'_*,$$

where

$$q = \bar{q} + q' \quad \text{and} \quad \psi = \bar{\psi} + \psi'.$$

In terms of  $\bar{\psi}_*$ ,  $\bar{q}_*$  is

$$\bar{q}_* = \frac{\bar{U}}{\beta L^2} \nabla_*^2 \psi_* + y_* + (F_* \psi_{*z_*})_{z_*},$$

where

$$z = \frac{N_0}{f} \left( \frac{\bar{U}}{\beta} \right)^{\frac{1}{2}} z_*, \quad F_* = N_0^2 N^{-2},$$

$N_0$  = typical value of the buoyancy frequency in the wind-driven gyre.

The vertical scale above was chosen to ensure that the vortex-stretching term and the  $\beta$ -effect have equal strengths. For a large-scale flow, away from western boundary layers, the non-dimensional parameter  $\bar{U}/\beta\bar{L}^2$  multiplying the relative vorticity is very small ( $\bar{U} \sim 1 \text{ cm s}^{-1}$ ,  $\beta \sim 10^{-13} \text{ cm}^{-1} \text{ s}^{-1}$ ,  $\bar{L} \sim 10^8 \text{ cm}$ , so  $\bar{U}/\beta\bar{L}^2 \sim 10^{-3}$ ). Thus in the interior of a wind-driven gyre one expects the relative vorticity to be negligible in comparison with the  $\beta$ -effect and vortex stretching. It is this approximation which ensures that horizontal momentum flux due to eddy stresses in the mean momentum equation is principally vertical (see §3.4 for further discussion).

For statistically steady flows driven externally by wind or buoyancy sources we write the time-average of (2.1) (non-dimensionalized):

$$\frac{\partial(\bar{\psi}_*, \bar{q}_*)}{\partial(x, y)} = -\epsilon \nabla \cdot \mathcal{F}_* + w_0(x, y) \delta(z) + \mu \frac{\partial}{\partial z_*} (F \nabla \cdot \mathcal{H}) + \nu \mathcal{D}, \quad (2.3)$$

where

$q_* = \bar{q}_*(\mathbf{x}, z) + q'_*(\mathbf{x}, z, t)$ ,  $\psi_* = \bar{\psi}_* + \psi'_*$ , ...,  $\mathcal{F}_* = \overline{q'_* \mathbf{u}'_*}$ ,  $\mathcal{H}$  is the buoyancy flux by deep convection or small-scale processes (including density diffusion),  $\mathcal{D}$  is the dissipation of potential vorticity by small-scale processes other than geostrophic eddies (e.g. internal waves),  $\epsilon = U'L'/\bar{U}\bar{L}$ ,  $\nu$  is the scale estimate of  $\mathcal{D}/\beta\bar{U}$ ,  $\mu = 1/H\beta\bar{U}$ ,  $w_0$  is the vertical velocity ( $\times f_0/\beta U$ ) produced by wind-stress curl at the base of the Ekman layer, and  $\delta(z)$  is the delta-function. The boundary conditions for insulating level upper and lower boundaries are

$$\frac{D}{Dt} \left( \frac{\partial \psi}{\partial z} \right) = 0 \quad (z = 0, -H), \quad \text{and} \quad \frac{\partial \psi}{\partial s} = 0$$

on a free-slip lateral boundary ( $s$  being the displacement along the boundary).  $\partial\psi/\partial z$  is proportional to perturbation density. Note the wind-stress forcing at the upper boundary has been shown explicitly on the right-hand side of (2.3) using the mathematical artifice of a delta-function.

The equations allow significant interaction of the upper boundary with the deep interior only by vortex stretching, but not by advection from the surface. A necessary condition for their validity is  $L/a \ll 1$ , where  $L$  is the lateral scale of a gyre streamline and  $a$  is the Earth's radius. It is just to this same degree that the mid-latitude  $\beta$ -plane is valid. Extension of the result to gyres of fully planetary breadth is straightforward so long as the contact of fluid with the upper and lower boundaries is infrequent in the sense described above. Further discussion of the density boundary condition is given by Rhines & Young (1982).

### 3. The integral constraint

In analysing a particular ocean-circulation model Holland & Rhines (1980) found it useful to study the vorticity balance integrated over an area bounded by a time-averaged streamline. In the uppermost layer, the circulation of the wind-stress,

$\rho_0 f_0 \iint w_0 dx dy = \oint \boldsymbol{\tau} \cdot d\mathbf{l}$ , was balanced by a combination of lateral momentum flux by eddies (to adjacent gyre streamlines) and downward momentum flux, which drove a deep circulation. In the deep layer the effect of this downward eddy flux was balanced principally by bottom friction. The procedure is analogous to zonal averaging in simplified atmospheric 'channel' models.

Here we use a similar integral balance to prove that, if gyres exist, (defined by curves  $\bar{\psi}$  or  $\bar{q} = \text{const}$  being closed upon themselves) then  $\bar{q}$  must in many cases be uniform within the gyres. There is an abrupt jump to the external  $q$ -field as one passes through the boundary of the wind gyre. If the exterior is at rest, this external field is just the 'ramp',  $q = \beta y$  (see figure 3b).

We give three separate derivations (§§ 3.1–3.3) of this result in order to distinguish necessary assumptions from those made for convenience. The first derivation closely parallels the argument given by Batchelor (1956); on a first reading it may be clearest to focus on this section and then move on to the remarks on vertical friction (§ 3.4) and atmospheric dynamics (§ 3.5).

### 3.1. Proof based on integrals round streamlines

The assumptions made in the following derivation are:

(i) The right-hand side of (2.3) is small, i.e.  $z < 0$  and  $\epsilon, \nu, \mu \ll 1$ . Thus as a first approximation the mean flow is dissipationless, and  $\bar{q}$  and  $\bar{\psi}$  contours are almost coincident as in (1.2).

(ii) The eddy-flux divergence has intermediate strength, i.e. based on the magnitudes of typical individual terms

$$J(\bar{\psi}, \bar{q}) \gg \epsilon \nabla \cdot \mathcal{F} \gg \nu \mathcal{D}, \quad \mu \frac{\partial}{\partial z} (F \nabla \cdot \mathcal{H}). \quad (3.1)$$

This includes assumption (i); the second inequality implies that the eddies are stronger than the process subsumed in  $\mathcal{D}$  and  $\mathcal{H}$  such as internal waves, penetrative convection, etc. Equation (3.1) suggests that the solution of (2.3) can be obtained as a perturbation expansion

$$\bar{q} = Q(\bar{\psi}, z) + \epsilon q_1 + \dots, \quad (3.2)$$

where the first term in the expansion is (1.2).

(iii) The eddy flux of potential vorticity can be approximated by

$$\begin{aligned} \mathcal{F}_i &= \overline{u_i' q'} \\ &= -\kappa_{ij} \frac{\partial \bar{q}}{\partial x_j} + O(\gamma), \end{aligned} \quad (3.3)$$

where  $\gamma = \frac{\text{eddy-particle excursion}}{\text{length scale of mean flow}} \ll 1$ ,

$\kappa_{ij}$  = Lagrangian diffusivity of fluid particles  
 = the ensemble average  $\langle u_i x_j \rangle$ , where  $x_i$  and  $u_i$  are the displacement and velocity of a particle.

The expression (3.3) has been discussed in detail elsewhere (Rhines 1977; Rhines & Holland 1979).

(iv) The symmetric part of  $\kappa_{ij}$ ,

$$S_{ij} = \frac{1}{2}(\kappa_{ij} + \kappa_{ji}), \quad (3.4)$$

is positive-definite, i.e.

$$S_{ij}a_i a_j > 0, \quad (3.5)$$

for all non-zero vectors  $a_i$ , everywhere in the flow. Note that the above is guaranteed if  $S_{ij} = A(\mathbf{x})\delta_{ij}$ . In general, however, the condition is

$$S_{11}S_{22} > S_{12}^2.$$

This condition on the symmetric part of  $\kappa_{ij}$  is related to the assertion that in a turbulent fluid a particle ensemble expands about its centre of mass, rather than contracts. Since

$$\begin{aligned} \mathcal{F} \cdot \nabla \bar{q} &= -\kappa_{ij} \frac{\partial \bar{q}}{\partial x_i} \frac{\partial \bar{q}}{\partial x_j} \\ &= -S_{ij} \frac{\partial \bar{q}}{\partial x_i} \frac{\partial \bar{q}}{\partial x_j}, \end{aligned} \quad (3.6)$$

(3.5) amounts to an assertion that the eddy flux of potential vorticity is downgradient everywhere. It will become apparent in the course of the derivation below that this last assumption is stronger than necessary; we will only require that (3.5) apply in an integrated sense.

Begin by integrating (2.3) over the area enclosed by a  $\bar{\psi}$ -contour. This area will be denoted by  $A_\psi$  and its boundary by  $B_\psi$ . The large advective term on the left-hand side vanishes identically leaving an integral balance between eddies, dissipation and heating:

$$-\epsilon \oint_{B_\psi} \mathcal{F} \cdot \hat{\mathbf{n}} dl + \nu \iint_{A_\psi} \mathcal{D} d^2a + \mu \oint_{B_\psi} (F\mathcal{H})_z \cdot \hat{\mathbf{n}} dl = 0 \quad (3.7)$$

(we take  $z \neq 0$ , so there is no contribution from the Ekman pumping  $w_0(x, y)$ ).  $\hat{\mathbf{n}}$  is a unit vector normal to  $B_\psi$  in the horizontal plane. Note that (3.7) is an exact result, not relying on assumptions (i)–(iv) above. It shows that the net potential vorticity flux across a closed mean streamline vanishes to  $O(\nu/\epsilon, \mu/\epsilon)$ .

Now from (3.1) and (3.3), (3.7) reduces to

$$\oint_{B_\psi} \kappa_{ij} \frac{\partial \bar{q}}{\partial x_j} \hat{n}_i dl = O\left(\frac{\nu, \mu}{\epsilon}\right) \ll 1,$$

and substituting the perturbation expansion (3.2) into the above, together with

$$\hat{\mathbf{n}} = \nabla \bar{\psi} / |\nabla \bar{\psi}|, \quad \nabla \bar{q} = \frac{\partial Q}{\partial \bar{\psi}} \nabla \bar{\psi} + O\left(\frac{\nu, \mu}{\epsilon}\right),$$

gives

$$\frac{\partial Q}{\partial \bar{\psi}} \oint_{B_\psi} S_{ij} \bar{\psi}_{,i} \bar{\psi}_{,j} \frac{dl}{|\nabla \bar{\psi}|} = 0. \quad (3.8)$$

Assumption (iv) then ensures that the line integral is non-zero, so that to leading order the potential vorticity is uniform. It is clear from (3.8) that (3.5) is stronger than necessary;  $S_{ij}$  need only be positive-definite in the integrated sense required by assuming that the line integral in (3.8) is non-zero. Since this notion is rather ill-defined we thought it best to make the strong assumption (3.5) from the outset.

This result provides a starting point for a theory of the general circulation (Rhines & Young 1982), driven by  $\tau$  and  $\mathcal{H}$ . It is evident that one feature of the circulation will be the occurrence of discontinuities in  $\bar{q}$ , for homogenization in gyres is so at odds with the  $\beta$ -gradient that must appear in resting fluid.



## 3.2. Proof based on integrals round geostrophic contours

The assumptions made in the derivation below are as follows.

(i) Closed geostrophic contours (i.e.  $\bar{q}$ -contours) exist. This is not obvious *a priori*; as has been mentioned previously the  $\beta$ -effect tends to prevent the closure of geostrophic contours. There are other more subtle processes which can prevent the closure of geostrophic contours. Suppose for example that the eddies are very strong in (2.3) so that  $\epsilon \gg 1$  and all the other terms are  $O(1)$  or less. Suppose furthermore that  $\kappa_{ij} = A\delta_{ij}$ , where  $A$  is constant, so that

$$\begin{aligned}\mathcal{F}_i &= -\kappa_{ij} \frac{\partial \bar{q}}{\partial x_j} \\ &= -A \frac{\partial \bar{q}}{\partial x_i},\end{aligned}$$

and consequently (2.3) reduces to

$$A\nabla^2 \bar{q} \simeq 0.$$

Hence, in this simplified strong-eddy limit,  $\bar{q}$  is harmonic and so can have no interior extrema. Consequently no closed  $\bar{q}$  contours exist.

(ii) The eddy-flux divergence is greater than the other terms on the right-hand side of (2.3) i.e.  $\epsilon \gg \mu, \nu$  and  $z < 0$ .

(iii) and (iv) These are the same as assumptions (iii) and (iv) of the previous derivation, that the turbulent flux of  $q$  is diffusive in nature.

The strongest difference between the proof in §3.1 and that here is in assumption (i); in this proof the weak-eddy assumption, which is necessary for the validity of (1.2), is not made. The new assumption is itself restrictive, but this version of the theory has some value. For example, in a numerical or field experiment, one has knowledge of the  $\bar{q}$ -field, which at large scale is simply  $f\rho_z/\rho$ .

The proof is now almost identical to that in §3.1. Integrate (2.3) over the area enclosed by  $q$ -contour to obtain

$$-\epsilon \oint_{B_q} \mathcal{F} \cdot \hat{\mathbf{n}} dl + \nu \iint_{A_q} \mathcal{D} d^2 a + \mu \oint_{B_q} (F\mathcal{H})_z \cdot \hat{\mathbf{n}} dl = 0, \quad (3.9)$$

where  $A_q$  denotes the area enclosed by the contour and  $B_q$  the bounding curve. As in (3.7) the large advective term has vanished identically leaving an integral balance. Once again (3.9) is exact, provided the area of integration exists! Now assumption (ii) reduces (3.9) to

$$\oint_B \mathcal{F} \cdot \hat{\mathbf{n}} dl = O\left(\frac{\nu, \mu}{\epsilon}\right). \quad (3.10)$$

Assumption (iii) and  $\hat{\mathbf{n}} = \nabla \bar{q} / |\nabla \bar{q}|$  can be used to rewrite the above as

$$\oint_{B_q} S_{ij} \frac{\partial \bar{q}}{\partial x_i} \frac{\partial \bar{q}}{\partial x_j} \frac{dl}{|\nabla \bar{q}|} = O\left(\frac{\nu, \mu}{\epsilon}\right).$$

It then follows from assumption (iv) that

$$|\nabla \bar{q}| = O(\nu/\epsilon, \mu/\epsilon) \ll 1,$$

i.e.  $\bar{q}$  is uniform to order  $(\nu \bar{L}/\epsilon, \mu \bar{L}/\epsilon)$ , where  $\bar{L}$  is the horizontal length scale of the mean flow.

## 3.3. Proof based on the enstrophy balance within a streamline

This third derivation of the potential-vorticity homogenization result is presented to show that one can avoid introducing the flux-gradient relation (3.3) as an intermediate step. Not surprisingly we shall find that the arguments and assumptions are similar to those used by Rhines (1977) and Rhines & Holland (1979) to justify (3.3) in the first place. They highlight the connection between the enstrophy cascade, systematic transport of  $q$  by eddies, and homogenization of  $\bar{q}$ .

The assumptions made in the derivation below are as follows.

(i) Once again we assume that the eddy-flux divergence in (2.3) has intermediate strength:

$$\bar{\mathbf{u}} \cdot \nabla \bar{q} \gg \epsilon \nabla \cdot \mathcal{F} \gg \nu \mathcal{D}, \quad \mu \frac{\partial}{\partial z} (F \nabla \cdot \mathcal{H}).$$

Thus the perturbation expansion (3.2) is appropriate, and (3.7) reduces to

$$\oint_{B\psi} \bar{\mathbf{u}}' q' \cdot \hat{\mathbf{n}} dl = O\left(\frac{\nu, \mu}{\epsilon}\right) \ll 1. \quad (3.11)$$

(ii) The second assumption concerns the balance of terms in the eddy enstrophy ( $q'^2$ ) equation. This result is derived from the fluctuation potential-vorticity equation

$$q_t' + \bar{\mathbf{u}} \cdot \nabla q' + \mathbf{u}' \cdot \nabla \bar{q} + \nabla \cdot \{\mathbf{u}' q' - \bar{\mathbf{u}}' q'\} = \Delta', \quad (3.12)$$

where  $\Delta'$  subsumes all the small-scale processes that dissipate enstrophy and so provide the end-point of the enstrophy cascade. (Equation (3.12) is obtained by subtracting (2.3) from (2.1).) If (3.12) is multiplied by  $q'$  and averaged one obtains the enstrophy equation

$$\bar{\mathbf{u}} \cdot \nabla \frac{1}{2} \overline{q'^2} + \bar{\mathbf{u}}' q' \cdot \nabla \bar{q} + \nabla \cdot \bar{\mathbf{u}}' \frac{1}{2} \overline{q'^2} = \overline{\Delta' q'}, \quad (3.13)$$

where we have assumed that the turbulence is statistically stationary so that  $\partial(\frac{1}{2} \overline{q'^2})/\partial t = 0$ . We further assume that the third term in (3.13) is negligible compared with the other three. Rhines (1977) and Rhines & Holland (1979) have argued that the circumstances in which this is valid are the same as those that justify (3.3), viz scale separation between the mean flow and eddy-induced particle excursions.

$$\iint_{A\psi} \bar{\mathbf{u}}' q' \cdot \nabla \bar{q} d^2a = \iint_{A\psi} \overline{\Delta' q'} d^2a + O(\gamma), \quad (3.14)$$

where  $\gamma$  is the scale-separation parameter defined after (3.3). Integrate the first term in (3.14) by parts:

$$\oint_{B\psi} \bar{q} \bar{\mathbf{u}}' q' \cdot \hat{\mathbf{n}} dl - \iint_{A\psi} \bar{q} \nabla \cdot \bar{\mathbf{u}}' q' d^2a = \iint_{A\psi} \overline{\Delta' q'} d^2a + O(\gamma). \quad (3.15)$$

Now both terms on the left-hand side of (3.15) are  $O(\nu/\epsilon)$  since

$$\begin{aligned} \oint_{B\psi} \bar{q} \bar{\mathbf{u}}' q' \cdot \hat{\mathbf{n}} dl &= Q(\bar{\psi}, z) \oint_{B\psi} \bar{\mathbf{u}}' q' \cdot \hat{\mathbf{n}} dl \quad (\text{from (3.2)}) \\ &= O\left(\frac{\nu, \mu}{\epsilon}\right), \quad (\text{from (3.11)}) \end{aligned} \quad (3.16)$$

and if (2.3) is multiplied by  $\bar{q}$  and integrated over  $A_\psi$  then

$$\iint_{A\psi} \bar{q} \nabla \cdot \bar{\mathbf{u}}' q' d^2a = O\left(\frac{\nu, \mu}{\epsilon}\right). \quad (3.17)$$

Hence (3.15) implies

$$\iint_{A\psi} \overline{\Delta'q'} d^2a = O\left(\frac{\nu}{\epsilon}, \frac{\mu}{\epsilon}, \gamma\right). \quad (3.18)$$

For particular forms of  $\Delta'$  such as

$$\Delta' = -\delta q' \quad \text{or} \quad \lambda \nabla^2 q' \quad (3.19)$$

(3.18) allows us to conclude that

$$q' = O\left(\frac{\nu}{\epsilon}, \frac{\mu}{\epsilon}, \gamma\right). \quad (3.20)$$

This, together with the observation that  $q'$  is created by displacing fluid particles across the mean potential vorticity gradient, implies that

$$\nabla \bar{q} = O\left(\frac{\nu}{\epsilon}, \frac{\mu}{\epsilon}, \gamma\right). \quad (3.21)$$

When  $\Delta'$  is not given by (3.19) (for example in many eddy-resolving general circulation models  $\Delta'$  is proportional to  $\nabla^4 \psi'$  or  $\nabla^6 \psi'$ ) one must argue that in an enstrophy cascade the average effect of small-scale dissipation  $\Delta'$  is to remove enstrophy so that  $\overline{\Delta'q'}$  is positive-definite and proportional to  $q'$ . The results (3.20) and (3.21) then follow. Rhines & Holland (1979) discuss the circumstances in which  $\overline{\Delta'q'}$  is negative and conclude that these exceptions are rare.

#### 3.4. Some remarks on vertical-friction models

In §1 we mentioned that when the mean flow is of large scale (more precisely when the relative vorticity does not contribute significantly to the mean potential vorticity field) the dominant mean effect of the quasi-geostrophic eddies in the mean momentum equation is vertical (rather than horizontal) transport of momentum. In this section we show the consequences of the familiar assumption that all mean effects of transient flows can be modelled by a simple viscosity acting on vertical gradients of velocity. Let

$$\frac{D\bar{\mathbf{u}}}{Dt} + \hat{\mathbf{z}} \times f\bar{\mathbf{u}} = -\nabla p/\rho_0 + (\nu \bar{\mathbf{u}}_z)_z. \quad (3.22)$$

In the above  $\bar{\mathbf{u}}$  is the mean horizontal velocity and  $\nu(z)$  is a kinematic vertical viscosity. The potential vorticity equation is obtained by taking the curl of (3.22) and then using the density equation (linearized about the mean stratification) to eliminate the vertical velocities. The result is

$$J(\bar{\psi}, \bar{q}) = -\nabla \cdot \mathcal{F}, \quad (3.23a)$$

$$\text{where} \quad \mathcal{F} = \hat{\mathbf{z}} \times (\nu \bar{\mathbf{u}}_z)_z, \quad \bar{q} = \beta y + (F \bar{f}_z)_z, \quad F = f_0^2/N^2(z). \quad (3.23b, c, d)$$

We are using the dimensional quasi-geostrophic equations; the relative vorticity has been neglected in (3.23c) using the scaling arguments of §2. The most important point to note is that the right-hand side of (3.23a) is not proportional to  $\nabla^2 \bar{q}$  unless  $\nu$  is proportional to  $F$ . Thus a lateral eddy flux of potential vorticity corresponds to vertical viscosity with a particular  $z$  dependence, viz

$$\nu(z) \propto F(z) \propto N^{-2}(z). \quad (3.24)$$

This choice of  $\nu(z)$  has implications for numerical modelling; it is to be preferred if one believes in (3.3).

Now suppose that the right-hand side of (3.23) is small, so that

$$\bar{q} = Q(\bar{\psi}, z) + O(\nu). \quad (3.25)$$

As in §§3.1–3.3, if (3.23) is integrated over  $A_\psi$  the large-left-hand side vanishes identically, leaving

$$\oint_{B_\psi} \mathcal{F} \cdot \hat{\mathbf{n}} \, dl = 0, \quad (3.26)$$

Once again (3.26) is an exact result; no assumptions such as  $\nu \ll 1$  have been made. However, using (3.23b) and (3.25), (3.26) can be rewritten as

$$\oint_{B_\psi} \mathcal{F} \cdot \hat{\mathbf{n}} \, dl = \oint_{B_\psi} (\nu \bar{\mathbf{u}}_z)_z \cdot \mathbf{dl} \quad (3.27a)$$

$$= \hat{\nu}_z \oint_{B_\psi} F \bar{\mathbf{u}}_z \cdot \mathbf{dl} + \hat{\nu} \oint_{B_\psi} (F \mathbf{u}_z)_z \cdot \mathbf{dl}, \quad (3.27b)$$

where we have introduced  $\hat{\nu}(z)$  defined by

$$\nu(z) = F(z) \hat{\nu}(z). \quad (3.28)$$

Using

$$\hat{\mathbf{z}} \times \nabla \bar{q} = \beta \hat{\mathbf{z}} \times \hat{\boldsymbol{\psi}} + (F \bar{\mathbf{u}}_z)_z, \quad (3.29a)$$

or

$$(F \bar{\mathbf{u}}_z)_z = \frac{\partial Q}{\partial \psi} \bar{\mathbf{u}} + \beta \hat{\mathbf{x}}, \quad (3.29b)$$

from (3.25), one can put (3.26) and (3.27b) in the form

$$\hat{\nu}_z \oint_{B_\psi} F \bar{\mathbf{u}}_z \cdot \mathbf{dl} + \hat{\nu} \frac{\partial Q}{\partial \psi} \oint_{B_\psi} \bar{\mathbf{u}} \cdot \mathbf{dl} + \hat{\nu} \beta \oint_{B_\psi} \hat{\mathbf{x}} \cdot \mathbf{dl} = 0. \quad (3.30)$$

If  $B_\psi$  is a mid-latitude streamline, which does not enclose the North or South Pole, then the third term in (3.30) is zero. Hence if  $\hat{\nu}_z \neq 0$  then the potential vorticity is not uniform:

$$\frac{\partial Q}{\partial \psi} = - \left( \frac{\nu}{F} \right)_z \frac{F^2 \oint \bar{\mathbf{u}}_z \cdot \mathbf{dl}}{\oint \bar{\mathbf{u}} \cdot \mathbf{dl}}.$$

On the other hand, when  $\hat{\nu}_z = 0$ , so that the vertical friction in (3.22) is analogous to that produced by eddy stresses and (3.23a) is

$$J(\bar{\psi}, \bar{q}) = \nu \nabla^2 \bar{q},$$

then the homogenization result is recovered.

The observation above that vertical friction must be specially chosen in order to homogenize potential vorticity is important because it allows one to discriminate between processes. For instance, if it is observed that the potential vorticity is indeed uniform in some part of the ocean one could argue that this homogenization was accomplished by lateral eddy flux of potential vorticity and not by some arbitrary vertical friction due to, say, internal waves. (Unless of course there was some reason for believing that vertical stress transmission by internal waves should be parameterized as in (3.24)!)

### 3.5. Atmospheric dynamics

Surfaces of constant potential density in an atmosphere free of lateral boundaries will tend to expel potential vorticity gradients. The homogenized state will, however, be

reached only if interior sources (e.g. radiative and latent heating) of  $q$  lying inside  $B_\psi$  or  $B_q$  are sufficiently weak. Also, regions where isentropic surfaces intersect the ground must not lead to significant flux of  $q$  if the result is to hold.†

The eddy flux of potential vorticity in (2.3) and the vertical friction in (3.22) may in this geometry lead to different mean circulations even when (3.24) is satisfied. Take the domain of integration to be the region north of a latitude  $y$ , on a constant potential density surface, still within the restrictions of the quasi-geostrophic equations. Then the argument in §3.1 applies when  $\mathcal{F} = \overline{\mathbf{u}'q'}$ , and so  $\bar{q}$  becomes nearly uniform, say  $\partial\bar{q}/\partial y = O(\nu/\epsilon)$ . On the other hand if  $\mathcal{F}$  is given by (3.23b) and if we choose  $\hat{\nu}_z = 0$ , then (3.30) becomes

$$\int_{B_\psi} \nabla\bar{q} \cdot \hat{\mathbf{n}} dl = -\beta \oint \hat{\mathbf{x}} \cdot d\mathbf{l} \\ = 2\pi a\beta \sin(\text{latitude}),$$

where  $a$  is the Earth's radius. If the circuit  $\bar{\psi} = \text{constant}$  coincides with a latitude circle this gives

$$\frac{\partial\bar{q}}{\partial y} = \beta, \quad \frac{\partial\bar{h}}{\partial y} = 0. \quad (3.31)$$

It is the thickness  $h$  of an isopycnal layer rather than the potential vorticity that is uniform.

Annular regions pose new problems. A popular idealization of the atmosphere is a re-entrant east-west running channel, with rigid walls at  $y = y_0$  and  $y = y_1$ . The circulation integrals have as their boundary circuits the contour  $\psi$  plus the northern or southern boundary. For such a  $\beta$ -plane channel with steady flow and vertical friction as in (3.22) with  $\hat{\nu}_z = 0$ , (3.30) becomes

$$\oint_{A_\psi} \nabla q \cdot \hat{\mathbf{n}} ds = \int \frac{\partial q}{\partial y}(x, y_1) dx.$$

Suppose for simplicity that  $\psi$  lies upon a latitude circle. Then  $q = q(y, z)$ , and from the above

$$\frac{\partial}{\partial y} \left( \frac{\partial q}{\partial y} \right) = 0. \quad (3.32)$$

To close the problem we must say something about the potential-vorticity flux through the rigid wall. If the flow vanishes in the neighbourhood of  $y_1$ ,  $(\partial q/\partial y)(x, y_1) = \beta$  and we recover  $\partial q/\partial y = \beta$ , as above. This is also the result if for any reason  $\mathcal{F}(y_1) = 0$ . But of more interest to atmospheric dynamicists are unsteady flows. The eddy closure (3.3) leads to (3.32). Given that the zonal momentum balance is

$$\frac{\partial \bar{u}}{\partial t} = \frac{f\bar{h}v}{H} + \overline{q'v'},$$

where  $h$  is the isopycnal-layer thickness, and  $H = \bar{h}$ , steady  $x$ -mean flows in a channel must obey  $\overline{q'v'} = 0$  if mass-conservation forces  $\bar{h}v = 0$ . This says simply that there is

† On the other hand homogenization is greatly enhanced by parallel-flow geometry, for which (3.2) is automatically satisfied. In the idealized 'atmosphere' of McWilliams & Chow (1981), absence of mean streamline curvature leads to nearly perfect homogenization on interior density levels.

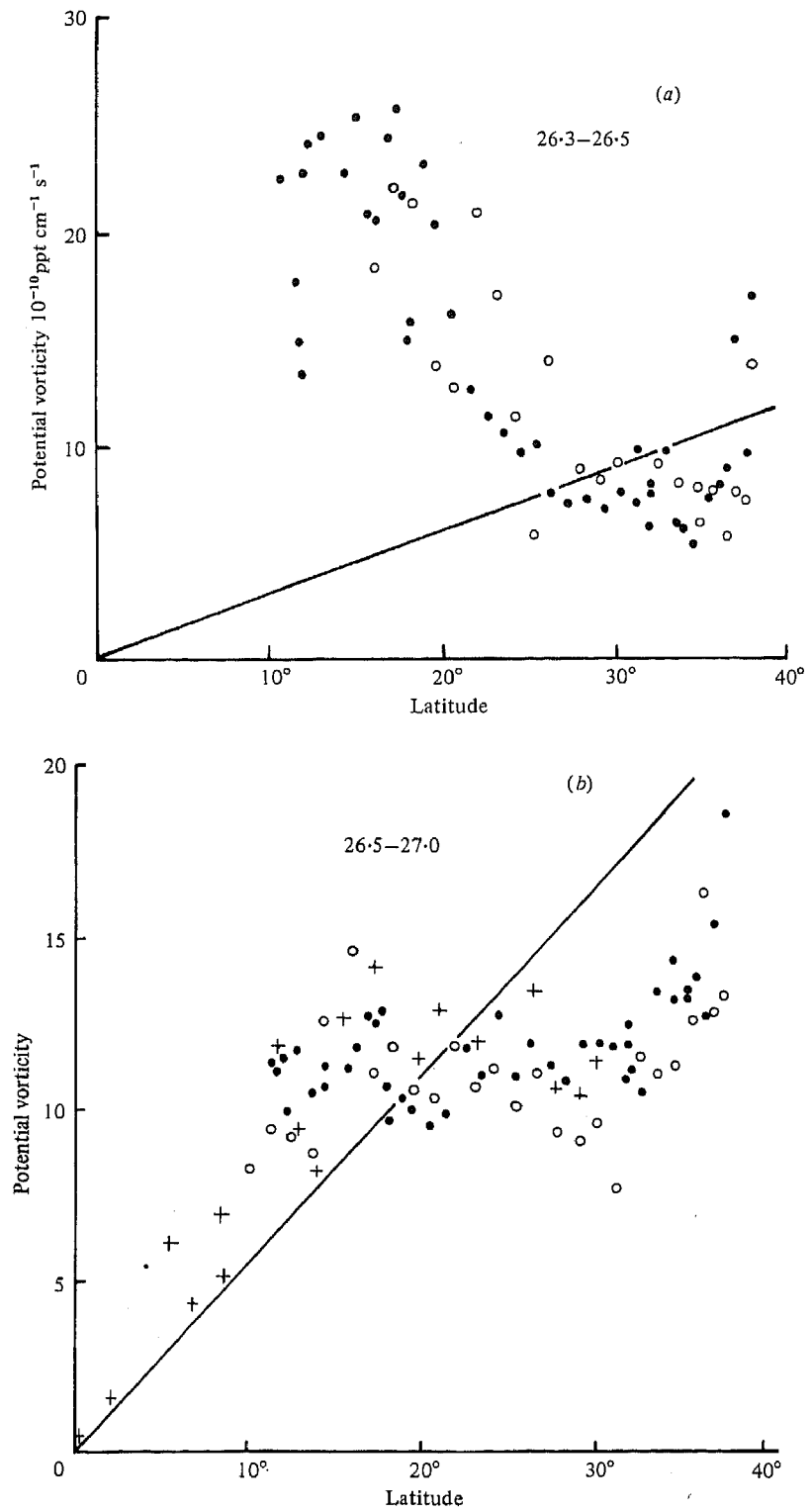


FIGURE 2. For caption see facing page.

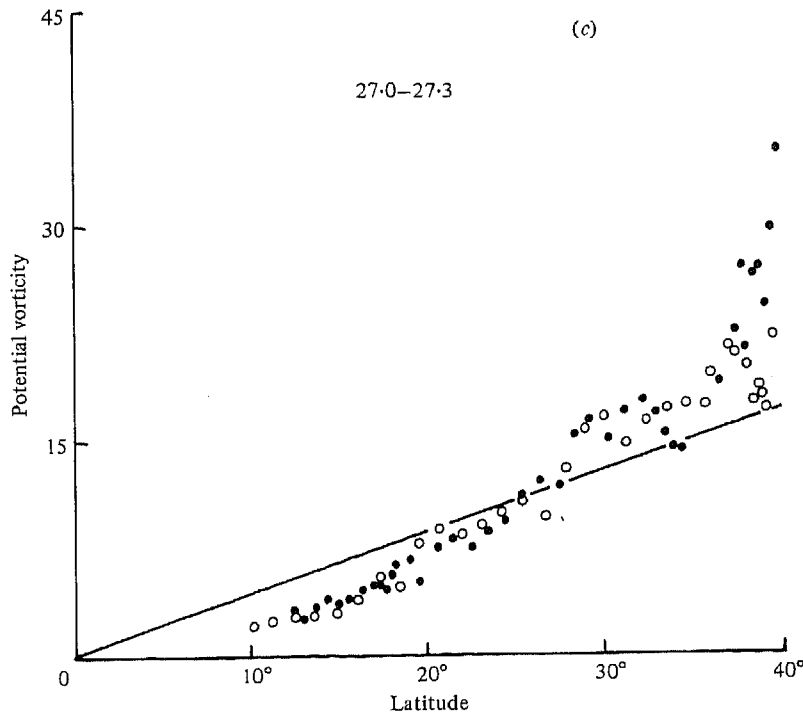


FIGURE 2. The large-scale potential vorticity  $q$ , as computed from individual (i.e. not time-averaged) hydrographic observations of the North Atlantic Ocean. (a) At the shallowest potential density surface,  $\sigma_\theta = 26.3-26.5$ , the potential-vorticity gradient is strongly negative as one moves into the wind gyre; (b)  $\sigma_\theta = 26.5-27.0$ , the  $q$ -gradient is approximately the ambient value,  $\beta/\bar{h}$  (plotted), south of  $10^\circ\text{N}$  latitude, but becomes much smaller than  $\beta$  inside the wind gyre; (c) at a level near the base of the wind gyre,  $\sigma_\theta = 27.0-27.3$ , the  $q$ -gradient is greater than  $\beta$ , and positive. It is the density interval (b) in which the theory predicts the expulsion of  $q$ -gradients. Interval (c) is near the level of vanishing velocity, and represents the rim of the gyre, where the potential vorticity gradient is concentrated. Data from International Geophysical Year cruises analysed by S. McDowell, data from Montgomery (1938) indicated with crosses.

nothing to balance the eddy force  $q'\nu'$ , thus  $\partial\bar{q}/\partial y = O(\nu/\epsilon)$ , as before. The homogenization of  $\bar{q}$  in a channel flow is the most relevant result of this section.

#### 4. Field observations and numerical simulations

This result emerged as a part of an analytical study of the general ocean circulation, which is presented in a succeeding paper (Rhines & Young 1982). The prediction  $\nabla\bar{q} = 0$  is in itself sufficiently interesting to stimulate us to look at data and computer models.

*North Atlantic Ocean.* Mr Scott McDowell (see McDowell *et al.* 1982) kindly computed the large-scale potential vorticity for the North Atlantic from a selected set of sections (many from the 1958 International Geophysical Year). At scales much larger than the Rossby radius of deformation appropriate to the observed stratification  $q$  is approximated well by  $\tilde{q} \equiv f/\bar{h}$ , or  $f(\partial\sigma_\theta/\partial z)/\sigma_\theta$ , where  $\sigma_\theta$  is the potential density.  $\tilde{q}$  in this approximation can be calculated from hydrographic observations alone. In figure 2 we plot values for  $q$  along north-south lines that cut across the subtropical wind-

driven gyre. The constant potential-density layers used for the calculations are  $\sigma_\theta = 26.3\text{--}26.5$ ,  $26.5\text{--}27.0$ , and  $27.0\text{--}27.3$ . The first of these (figure 2*a*) is deepest in the gyre centre (about 400 m) where it is 130 m thick. It intersects the sea surface in the north-eastern part of the gyre. The second, deeper surface extends to 800 m depth in mid-gyre and has a typical thickness of 400 m. Its potential temperature ranges from  $11^\circ$  to  $17^\circ\text{C}$ . The third, deepest surface lies below the dominant flow of the wind gyre, reaching to 1000 m in mid-gyre, with a thickness of about 200 m.

The shallowest of the three layers intersects regions of direct atmospheric forcing (where winter cooling and evaporation cause deep convection in the  $18^\circ$  water formation area at the northern edge of the Sargasso Sea), and regions where boundary-current dissipation may be strong. The conditions for validity of the theorem are not met, and indeed there is a sharp decrease in  $\tilde{q}$  as one moves northward. The potential vorticity  $\beta y/\bar{h}$  appropriate to a resting ocean is plotted for comparison.

In the intermediate-depth layer the conditions of the derivation are satisfied, and a large region of essentially uniform potential vorticity shows through the observation (and sampling) noise (see figure 2*b*). The northern and southern boundaries of the gyre are evident where large gradients of  $\tilde{q}$  appear. These boundaries are well-documented by other markers like salinity, dissolved oxygen, tritium, and by velocity measurements. The I.G.Y. data agrees well with that of Montgomery (1938). The most misleading part of the plot occurs in the Gulf Stream, which forms the northern border of the gyre. There the relative vorticity forms an important part of  $q$ , which may be far more uniform across the Stream than the  $\tilde{q}$ -sections suggest.

The deepest of the three levels is at the base of the wind-driven gyre. As expected, strong gradients of  $\tilde{q}$  appear. These north-south profiles of  $\tilde{q}$  correspond simply to the geometry of the isopycnal surfaces sketched in figure 1. The dish-shaped depression of the surfaces decreases at greater depth, and the dips are shifted poleward. This amounts to a thickening of isopycnal layers as one moves northward. The thickening cancels  $\beta y$  in the expression for  $q$ , over a large range of depths. Below these levels the gradient of  $q$  must be made up, and is thus doubly strong. Above the homogenized layers the northward thickening of the layers overcompensated  $\beta$ , presumably because of sources and sinks of potential vorticity at shallow levels within gyre contours.

The strong potential vorticity gradients at the base of the gyre make the map of  $\tilde{q}$ , and hence the flow direction, particularly well-defined at these levels. But, ironically, it is just these levels where the theory suggests the flow is becoming weak and indistinct. The circulation at these great depths is more readily driven by direct forcing, for example by buoyancy sources and diffusion. The character of a non-diffusive thermaline circulation in the deep water should require 'tongues' of  $q$  aligned with the flow, rather than the completely homogenized  $q$  of the wind gyres.

In a series of papers (e.g. Stommel & Schott 1977; Behringer & Stommel 1980) the  $\tilde{q}$ -field in a  $10^\circ$  triangle in the eastern North Atlantic has been explored intensively. This amounts to a local analysis of the function  $q(\bar{\psi}, z)$ . The object was to infer the flow direction from the strike of the geostrophic contours, and hence to find the constant of integration in the determination of velocity from density observations. In agreement with McDowell's computations, the layer at 600–800 m depth has well-defined  $\tilde{q}$ -contours, and the layer above 450 m depth has unresolved, small gradients of  $q$ . As we infer above the vector velocity vanishes roughly at 750 m depth, in the heart of the strong  $q$ -gradient.



*Numerical models: subtropical wind gyre.* Few numerical simulations of the ocean have operated in the weakly diffusive parameter range appropriate to the theory (and, perhaps, to the oceans). Islands and plateaux of  $q$  develop most clearly at levels out of reach of the direct influence of the wind and above the direct influence of the bottom frictional layer. A two-layer model fails to harbour any level at which the theory applies accurately. Nevertheless, the expulsion effect is sufficiently strong that in the simulation discussed by Holland & Rhines (1980) regions of quasi-uniform  $\bar{q}$  appeared in both layers. Coincident with the development of this theory, Holland has produced a new three-layer  $4000 \times 4000$  km model of the wind-driven circulation. In the crucial interior layer both  $\bar{q}$  and the instantaneous  $q$  vanish in a region  $4000 \times 3000$  km in extent (figure 3). The homogenized region coincides with the pair of wind-driven gyres which are antisymmetric about the middle latitude of the basin. It is interesting that the western boundary currents and the intense free jet that flows eastward along the line of symmetry are completely invisible in the  $q$ -field. The expulsion of  $q$ -gradients has been this complete. North-south sections of  $\bar{q}$  in each of the three layers resemble the North Atlantic data shown in figure 2.

The argument leading to (3.24) suggests that, if eddies are neglected in a coarsely resolved numerical simulation, a vertical Navier-Stokes friction  $\nu(z)$  proportional to buoyancy  $N^2$  will reproduce some of their effect on the mean state.

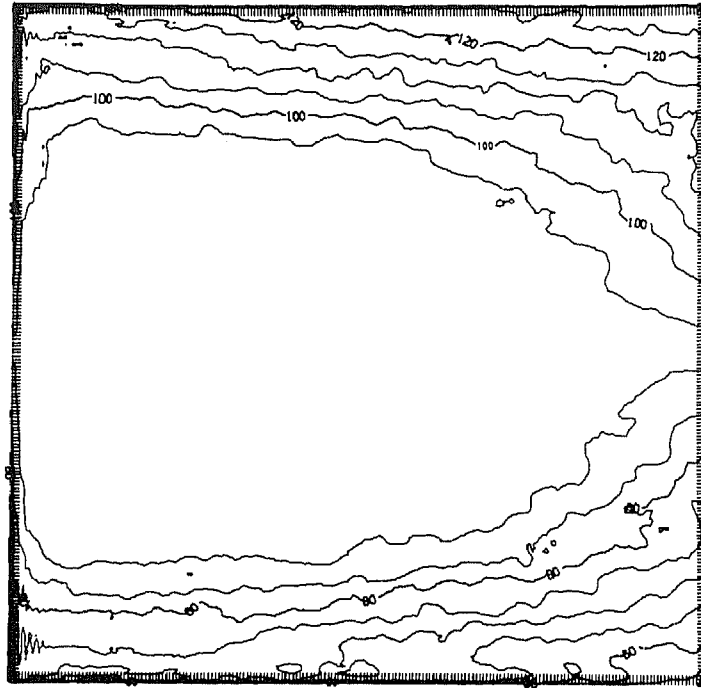
*Wind-driven zonally oriented channel.* McWilliams & Chow (1981) have recently added an interior layer to their comprehensive model of turbulent  $\beta$ -plane channel flows (McWilliams, Holland & Chow 1978). In this new interior layer the time- and zonally averaged potential vorticity is uniform to within a few parts per thousand. This suggests that eddies indeed act more like a down-gradient lateral diffusion of  $q$  (3.3), in appropriate circumstances, than a down-gradient vertical momentum diffusion (3.23*b*).

Once expulsion has occurred to this degree, the homogenized layers are 'slaved' to the 'outer' layers, in which  $q$  is not uniform. The entire flow, that is, may be found from the  $q$ -field in the outer layers alone. This suggests that further numerical computation might be simplified by exploiting the reduction in degrees of freedom.

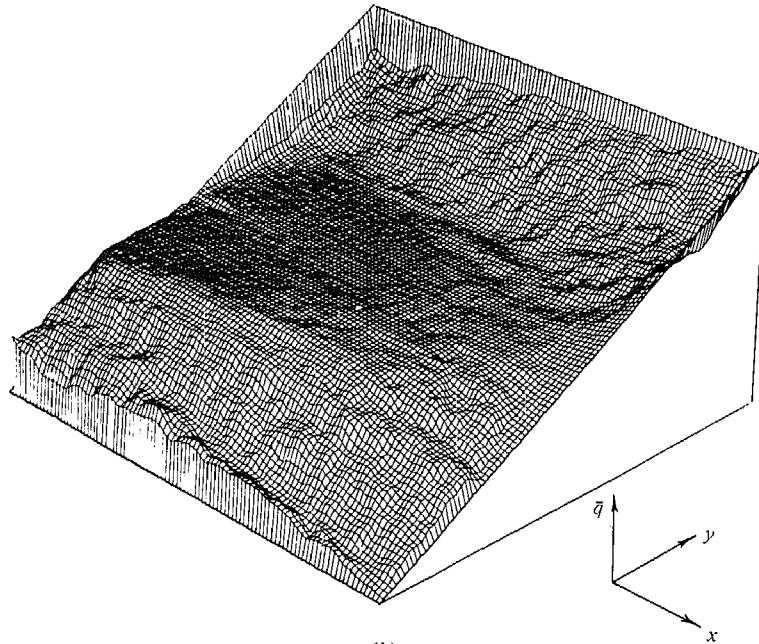
## 5. Concluding remarks

We have suggested that planetary gyres driven remotely (i.e. across surfaces of constant potential density rather than along them) tend, firstly, to involve flow nearly along geostrophic contours in regions of weak eddies, and, secondly, to expel gradients of potential vorticity to the outer edges of the gyres. It is in this manner that the otherwise dominant effect of planetary sphericity (the  $\beta$ -effect) is defeated and north-south excursions of fluid particles are made possible. The depth range of the wind-driven ocean gyres is obviously limited by the necessity that the flow 'burrow out' plateaux of uniform  $q$ ; the extent will depend on the strength of the wind stress. This is the topic of the succeeding theory (Rhines & Young 1982).

The rate at which expulsion occurs is important, for this effect must compete with others in determination of  $Q(\psi)$ . In a paper in preparation we show that the fast initial phase of expulsion relies on the process known as shear dispersion, which replaces initial values of  $q$  by their (generalized) average about a closed streamline. This phase requires more than the circulation time  $L/U$  (where  $L$  is the scale of the flow), yet less than the diffusion time  $L^2/\kappa$ . For steady flow with small  $\kappa/UL$  this time is of order



(a)



(b)

FIGURE 3. For caption see facing page.

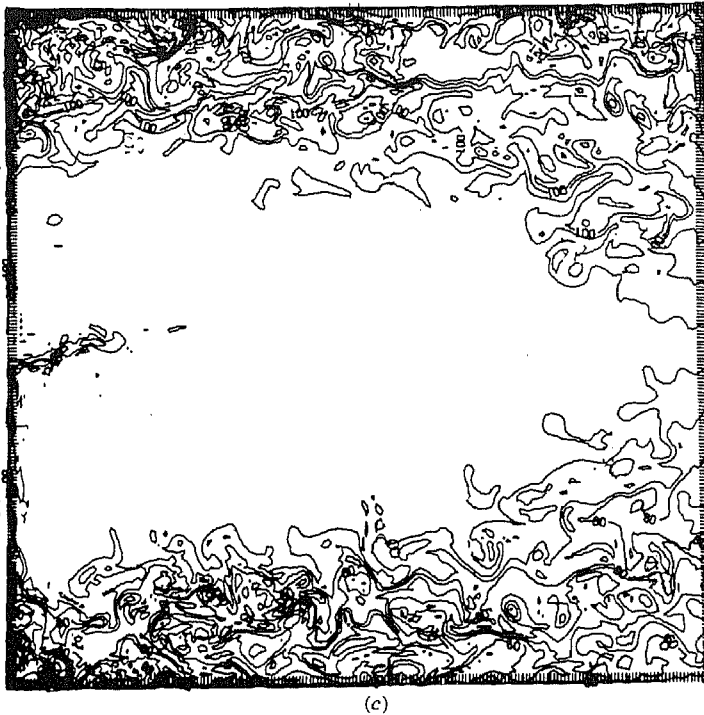


FIGURE 3. The potential-vorticity field from a numerical simulation of the wind-driven circulation by W. Holland. This is the intermediate layer (300–1000 m depth) of the 5000 m  $\times$  4000 km  $\times$  4000 km basin. Resolution of the grid is 20 km, with three layers in the vertical. Driving is by a steady east–west wind stress exerted on the uppermost layer. The simulation develops a pair of wind gyres antisymmetric about the middle latitude, plus violent eddy activity. Intense western boundary currents occur with a free jet along the middle latitude of the basin. The time-averaged potential vorticity (*a*) is nearly uniform in the domain of the circulation gyres, in accord with the theory. Outside the gyres the planetary gradient of  $\bar{q}$  is visible. (*b*) A perspective view of the  $\bar{q}$ -field. (*c*) The instantaneous  $q$ -field shows nearly perfect homogenization, within the pair of gyres. At the edges the level of eddy activity is clear, with  $q$ -contours being wound up in the enstrophy cascade. Occasional wisps of anomalous  $q$  are swept along the western boundary and ejected into the middle of the domain, where they are rapidly dissipated.

$(L/U)(UL/\kappa)^{\frac{1}{2}}$ . The second, slow phase of expulsion is entirely diffusive, with any residual variation of  $q$  across streamlines disappearing over a time  $\sim L^2/\kappa$ . The case of solid-body rotation is special: there is no shear-augmented dispersion and the entire process requires the long diffusive time  $\sim L^2/\kappa$ .

The theory given here does not fully come to grips with western boundary current dynamics, in which the dissipation may be large. Treatment of this area is in progress. It is clear that, given sufficiently slippery lateral boundaries, the homogenized gyres can still exist. † Both the oceanic observations and the numerical simulations described above support the picture of gyres in which interior layers circulate without great

† The subtlety of the boundary effects must be emphasized. Current numerical models using ‘high-order’ friction (a term  $\nabla^6\psi$  in the vorticity equation) require additional boundary conditions, beyond just the vanishing of velocity at a rigid wall, or vanishing normal velocity and stress at a ‘slippery’ wall. The additional conditions chosen will affect the flux of potential vorticity through the wall, and hence the degree of homogenization of gyres.

change of their potential vorticity, even in boundary currents. The contrast with one's intuition based on homogeneous fluid models of the circulation is great. There, large changes in  $q$  are caused by passage of fluid through a dissipative Gulf Stream boundary current, changes which are removed by the wind-stress curl in mid-ocean. Dissipation of  $q$  is necessary in all cases, but in the present model this can occur in the exterior density surfaces. The boundary current occurs in interior levels not by necessity of dissipation, but merely to mimic the  $\bar{q}$  and eddy-stress patterns imposed upon them from the exterior layer, above.

The Lagrangian history of fluid particles will be an important diagnostic tool for these dynamics. Fluid particles diverge from Eulerian-mean streamlines by the process of eddy dispersion (the very quantity described by  $\kappa_{ij}$  in (3.3)). Stirring and eventual mixing by molecular effects causes the fluid eventually to depart from surfaces of constant potential density and potential vorticity. The present theory assumes that this migration process is slow relative to the circulatory nature of the gyres (that is, the pitch of the 'helical' particle paths is tight), and that horizontal scattering of particles from one gyre to the next is gradual. The potential vorticity balance following particles should be simple, indeed, in homogenized gyres. Not only is the time-average of  $q$  uniform, but so too is the *instantaneous* field of  $q$ , by virtue of the transport argument!

The near vanishing of eddy flux of  $q$  that this implies leads to an apparent paradox for studies of the maintenance of the circulation. One might wrongly regard the effect of eddies to be weak, though they entirely determine the circulation. It is very like a turbulent Couette flow in which the Reynolds stress actively transports momentum yet the transport is non-divergent. The situation amounts to a singular limit of formulas relating the mean flow to eddy activity. For example Rhines (1977) and Yamagata (1981) give  $\bar{u} = -\overline{q'^2}/\nabla\bar{q}$  for a homogeneous-density geostrophic flow with bottom friction. When homogenization of  $q$  occurs, both numerator  $\overline{q'^2}$  and denominator  $\nabla\bar{q}$  vanish.

Our de-emphasis of the atmosphere stems from lack of appropriate observational data. In large regions internal thermal sources and sinks are active, and the theory will not apply there. Computations of the appropriate Ertel potential vorticity are under way by Held, Lau and Wallace, and their results are awaited with interest. Rossby (1947) indeed suggested that the circulation poleward of the jet stream was sufficiently stirred to make the vertically integrated potential vorticity uniform.

The authors are grateful to S. McDowell for computing the North Atlantic potential vorticity, which will appear in more detail in his Ph.D. dissertation. Drs Holland and McWilliams kindly made available their computer simulations before publication. We thank the referees for comments on presentation, and, equally, Mary Ann Lucas for accommodating the many changes. This work was supported by the National Science Foundation, POLYMODE grant OCE 78-25692 and OCE 75-21674, by the John Simon Guggenheim Foundation, and by Christ's College, Cambridge. The hospitality of the Department of Applied Mathematics and Theoretical Physics at Cambridge is gratefully acknowledged. This is contribution number 4848 of the Woods Hole Oceanographic Institution.

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