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Rossby Wave Action, Enstrophy and Energy in Forced Mean Flows†

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Assuming there is a separation in scale between the mean flow and fluctuations, the linearized potential vorticity equation is solved using the WKB method. Attention is focused on wave properties such as action and enstrophy which in some circumstances are conserved. In the most general case of Rossby waves supported by an arbitrary mean potential vorticity field, $\bar{q}=f/h$, and propagating through a forced mean flow neither action nor enstrophy is conserved. It is shown that action is produced by the forcing which drives mean flow across $\bar{q}$ contours, while enstrophy is produced both by complicated $\bar{q}$ contours and by horizontal divergence of the mean flow.

1. INTRODUCTION

The interaction of Rossby waves with zonal mean flow has been extensively studied (see Dickinson, 1978, for a review). The energy density E of a Rossby wave train on a β-plane is not conserved as it propagates through a slowly varying mean flow. Instead, if the mean flow is zonal (i.e. unforced), the action density $A=\hat{\omega}^{-1}E$ defined by Bretherton and Garrett (1968) is conserved,

$$\partial A/\partial t+\nabla\cdot(\mathbf{C}A)=0, \tag{1}$$

where $\mathbf{C}$ is the group velocity and $\hat{\omega}$ the intrinsic frequency.

If the mean flow is forced the problem is more complicated. Müller (1978) proved that A is not conserved by waves propagating through a slowly varying, forced mean flow on a homogeneous, constant depth, β-

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plane ocean. However, it is shown below that in this case the enstrophy density of the wave packet,

$$\begin{aligned} P &= (k^2 + l^2)E \\ &= -\beta k A, \end{aligned} \tag{2}$$

is conserved and that the analogous wave-potential enstrophy is conserved in a stratified, forced flow. When the mean flow is independent of x, k (the x-wavenumber) is constant and P is proportional to A.

The purpose of this investigation is to derive the equations governing the change of quadratic wave properties such as E, A and P in the general case of Rossby waves propagating through a forced mean flow in an ocean with slow depth variation. In particular our results are relevant in the gently forced interior of a homogeneous ocean where the Sverdrup balance for the mean flow $(\bar{u}, \bar{v})$ with depth $h(x, y)$,

$$\bar{u}\bar{q}_x + \bar{v}\bar{q}_y = F, \tag{3}$$

$$\bar{q} = [f_0 + \beta y]/h(x, y), \tag{4}$$

obtains. As will be seen in Section 4 depth variations introduce several complications; in Section 2 we discuss the simpler problem of Rossby waves propagating vertically through a stratified incompressible fluid. In Section 3 a simple example illustrating the nonconservation of action in a forced mean flow is given.

2. ROSSBY WAVE TRAINS IN THREE DIMENSIONS

On a mid-latitude β-plane the linearized perturbation geostrophic potential vorticity equation in the Boussinesq approximation (see, e.g. Holton, 1975) is

$$q'_t - \bar{\psi}_y q'_x + \bar{\psi}_x q'_y - \psi'_y \bar{q}_x + \psi'_x \bar{q}_y = 0, \tag{5}$$

where

$$q' = \psi'_{xx} + \psi'_{yy} + (f_0^2 N^{-2} \psi'_z)_z,$$

$$\bar{q} = \bar{\psi}_{xx} + \bar{\psi}_{yy} + (f_0^2 N^{-2} \bar{\psi}_z)_z + \beta y.$$

Assume that there is a separation in scale between the mean flow and the perturbations and look for a solution of (5) using the WKB ansatz

$$\psi' = a(X, Y, Z, T) \exp\{i\mu^{-1}\theta(X, Y, Z, T)\}. \tag{6}$$

where

$$(X, Y, Z, T) = \mu(x, y, z, t),$$

and

$$\mu = \frac{\text{Length (or time) scale of perturbations}}{\text{Length (or time) scale of waves}} \ll 1.$$

Equation (6) is substituted into (5) and equal powers of μ collected to produce the hierarchy (dropping capitals)

$$\mu^0: \hat{\omega}(k^2 + l^2 + f_0^2 N^{-2} m^2) + \beta k = 0, \tag{7}$$

$$\mu^1: \left(\frac{\partial}{\partial t} - \bar{\psi}_y \frac{\partial}{\partial x} + \bar{\psi}_x \frac{\partial}{\partial y}\right)[(k^2 + l^2 + f_0^2 N^{-2} m^2)a]$$
$$-2\hat{\omega}\mathbf{K} \cdot \nabla a - \hat{\omega} a \nabla \cdot \mathbf{K} - \beta a_x = 0, \tag{8}$$

where

$$(k, l, m, \omega) = (\theta_X, \theta_Y, \theta_Z, -\theta_T),$$

and

$$\mathbf{K} = (k, l, f_0^2 N^{-2} m).$$

(It has been assumed that the Brunt–Väisälä frequency N varies on the same scale as the mean flow.) Eq. (7) is just the dispersion relation

$$\hat{\omega} = \omega + \bar{\psi}_y k - \bar{\psi}_x l = -\beta k/(k^2 + l^2 + f_0^2 N^{-2} m^2).$$

Eq. (8) describes the variation in amplitude of the wave packet, after a little algebra it can be put in the more intuitive form

$$\partial E/\partial t + \nabla \cdot (\mathbf{C}E) = \tfrac{1}{2} a^2 K_i K_j \bar{v}_{i,j} + \tfrac{1}{2} a^2 K_3 K_i \partial \bar{v}_i/\partial z,$$
$$E = \tfrac{1}{4}(k^2 + l^2 + f_0^2 N^{-2} m^2) a^2,$$
$$(\bar{v}_1, \bar{v}_2) = (-\bar{\psi}_y, \bar{\psi}_x) = \left\{\begin{array}{l}\text{the geostrophic part of}\\ \text{the mean velocity field,}\end{array}\right. \tag{9}$$

where i and j equal 1 and 2. The first term on the right-hand side of (9) is the conversion of mean kinetic energy to E by horizontal Reynolds stresses while the second term is the conversion due to vertical buoyancy flux. The derivation of (9) from the basic equations is given in Appendix A.

Surprisingly, the energy conversion terms on the right-hand side of (9) can be further simplified using the standard expressions for the rate of change of wavenumber along a packet trajectory (Lighthill, 1978)

$$\frac{dk}{dt} = -k\frac{\partial v_1}{\partial x} - l\frac{\partial v_2}{\partial x} - \frac{\partial \hat{\omega}}{\partial x}, \tag{10}$$

with analogous expressions for l and m. Since $\hat{\omega}$ has no explicit x or y dependence it follows that

$$\frac{d}{dt}(K_i^2) = -2K_iK_j\bar{v}_{i,j}, \quad i,j=1,2,$$

$$\frac{d}{dt}(f_0^2N^{-2}m^2) = -2K_3K_i\left(\frac{\partial \bar{v}_i}{\partial z}\right),$$

and so (9) can be rewritten as

$$\partial P/\partial t + \nabla \cdot (\mathbf{C}P) = 0,$$

$$P = (k^2 + l^2 + f_0^2N^{-2}m^2)E. \tag{11}$$

Note that since

$$P = -\beta k A,$$

it follows that

$$\partial A/\partial t + \nabla \cdot (\mathbf{C}A) = -A(d/dt)(\ln k)$$

$$= k^{-1}AK_i\partial\bar{v}_i/\partial x, \quad i=1,2;$$

A is conserved when the mean flow is unforced.

Integrating (11) over a volume which properly contains the wave train one finds

$$\frac{\partial}{\partial t}\int P\,dv = 0, \tag{12}$$

so that the total enstrophy is conserved. It is instructive to derive this result directly from (5). Multiply (5) by q' and average over a period to obtain

$$(\partial/\partial t)(\tfrac{1}{2}\overline{q'^2}) + \nabla \cdot (\bar{\mathbf{v}}\tfrac{1}{2}\overline{q'^2}) + \overline{q'\mathbf{v}'} \cdot \nabla\bar{q} = 0. \tag{13}$$

The crucial scale separation assumption implies

$$\nabla \bar{q} = \beta \hat{\mathbf{y}} + O(\mu^2),$$

so that (13) simplifies to

$$(\partial/\partial t)(\tfrac{1}{2}\overline{q'^2}) + \nabla \cdot (\bar{\mathbf{v}}\tfrac{1}{2}\overline{q'^2}) + \beta\overline{q'v'} = 0. \tag{14}$$

Integrating (14) over a large volume containing the train reproduces (12). This derivation emphasises the importance of the scale separation assumption which ensures that $\Delta\bar{q}$ is constant over the wave train. This restriction is also inherent in the WKB derivation, note how (7) and (8) are unchanged if $\bar{q}$ is simply taken to be βy. This does not mean that the shear in the mean flow has been completely neglected; from (9) the WKB approximation accounts for the energy conversion associated with mean shear.

3. AN EXAMPLE OF NONCONSERVATION OF ACTION

As a concrete example of nonconservation of wave action (but conservation of wave enstrophy) consider Rossby waves superimposed on a *meridional* flow in a homogeneous, constant depth ocean. Geisler and Dickinson (1975) analysed the critical level absorption of Rossby waves in such a flow. Because the fluid is homogeneous we can employ conservation of barotropic potential vorticity (see Appendix B) rather than the less exact conservation of geostrophic potential vorticity used in Section 2.

Since the mean flow is meridional the linearized potential vorticity equation is

$$\left[\frac{\partial}{\partial t} + \bar{v}(\mu x)\frac{\partial}{\partial y}\right]\nabla_2^2\psi' + \beta\psi'_x - \bar{v}_{xx}\psi'_y = 0. \tag{15}$$

The coefficients of (15) are independent of y and t, so a solution can be found in the form

$$\begin{aligned} \psi' &= \varphi(X)\exp i(ly - \omega t), \\ X &= \mu x, \end{aligned} \tag{16}$$

where ω and l are constants and φ satisfies

$$\left\{\left(\bar{v} - \frac{\omega}{l}\right)\left(\mu^2\frac{d^2}{dX^2} - l^2\right) - \frac{i\beta}{l}\frac{d}{dX} - \mu^2\frac{d^2\bar{v}}{dX^2}\right\}\varphi = 0. \tag{17}$$

The WKB solution of (17) is (see, e.g., Bender and Orszag, 1978)

$$\varphi_{1,2}=[(\beta/2\hat{\omega})^2]^{-1/4}k_{1,2}^{-1/2}\exp\left[i\mu^{-1}\int^{X}k_{1,2}\,dX\right], \tag{18}$$

where k_1 and k_2 are the solutions of the quadratic equation

$$\hat{\omega}=\omega-l\bar{v}(X)=-\beta k(k^2+l^2)^{-1}. \tag{19}$$

For a linear shear, $\omega-l\bar{v}=\alpha X$, k_1 and k_2 are plotted in Figure 1. From (18) and (19) it follows that

$$A(X)=\hat{\omega}^{-1}E=-\tfrac{1}{4}(\beta kl)^{-1}(k^2+l^2)^2, \tag{20}$$

so that

$$C_xA \text{ is proportional to } k^{-1}(X). \tag{21}$$

i.e. action is not conserved [cf. (1)] but

$$P=-\beta kA$$

is conserved. This can be deduced from the more general results of Section 2: simply suppress the term $f_0^2N^{-2}m^2$.

It is interesting to solve the ray tracing problem for a wave packet in the linear shear $\omega-l\bar{v}=\alpha X$; the ray equations are (Lighthill, 1978)

$$\frac{dk}{dt}=-\frac{\partial\omega}{\partial X},$$

so

$$k=k_0-\alpha t,$$

and

$$\frac{dx}{dt}=\frac{\partial\omega}{\partial k},$$

so

$$\frac{1}{X}-\frac{1}{X_0}=\frac{\alpha^2}{\beta}t+\frac{\alpha l^2}{\beta}\left(\frac{1}{k}-\frac{1}{k_0}\right).$$

The x wavenumber decreases linearly with time. A wave packet which starts at A on Figure 1 moves East initially, is reflected at B, passes through the critical layer at C unscathed (Geisler and Dickinson, 1975), is reflected again at D and is finally absorbed at the critical layer near E.

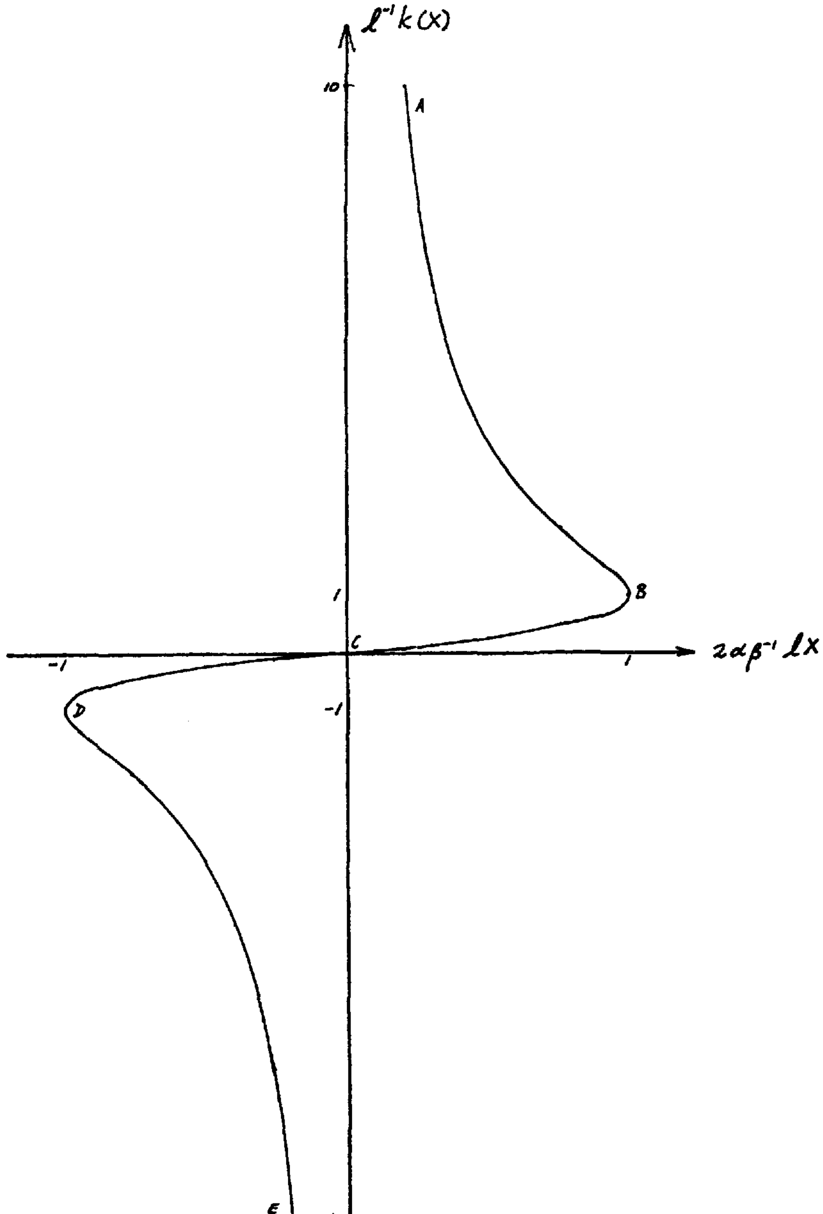

FIGURE 1 The solutions of (19) when $\hat{\omega}=\alpha X$. For each value of X there are two X wavenumbers; the waves on DCB have group velocity directed Westward while those on AB and DE have Eastward pointing group velocities. The critical layer is at $x=0$, as explained in Geisler and Dickinson (1975) only the short Eastward travelling waves suffer critical layer absorption.

The WKB solution (18) is, of course, invalid at the turning points and the critical layer where (18) is singular.

4. ROSSBY WAVE TRAINS IN AN OCEAN OF VARYING DEPTH

In Section 2 we considered waves in a stratified fluid and used conservation of geostrophic potential vorticity. In this section we discuss waves in a homogeneous fluid and use the more exact conservation of barotropic potential vorticity (see Appendix B),

$$\partial q'/\partial t + \bar{\mathbf{v}}\cdot\nabla_2 q' + \mathbf{v}'\cdot\nabla_2\bar{q} = -q'h^{-1}S, \tag{22}$$

where

$$q' = \zeta'/h, \quad \bar{q} = (f/h) + O(\mu^2), \tag{23, 24}$$

$$\nabla_2\cdot(h\mathbf{v}') = 0, \;\; \nabla_2\cdot(h\bar{\mathbf{v}}) = S, \tag{25, 26}$$

and

$$\nabla_2 = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}.$$

The fluid source S in (26) is produced by the wind stress curl which pumps fluid out of the upper Ekman layer into the interior of the ocean. This is the forcing mechanism which gently drives mean flow across $\bar{q}$ contours according to the classical Sverdrup balance (see Appendix B)

$$\bar{\mathbf{v}}\cdot\nabla_2\bar{q} = h^{-1}\{\nabla\times\mathbf{M}\cdot\hat{\mathbf{z}} - \bar{q}S\}. \tag{27}$$

The mean forcing term S appears in the perturbation vorticity equation (22). This is in contradistinction to (5) where mean forcing, such as diabatic heating and mechanical stress $\mathbf{M}$, appears only in the mean vorticity equation (3). Thus $\mathbf{M}$ and S are not equivalent, S can produce perturbation enstrophy (e.g. Appendix C) but $\mathbf{M}$ cannot.‡

Because of (25), we can introduce a mass streamfunction

$$h\mathbf{v}' = \hat{\mathbf{z}}\times\nabla\psi', \tag{28}$$

and

$$q' = h^{-1}\zeta' = h^{-2}\{\nabla_2^2\psi' - \nabla_2\ln h\cdot\nabla_2\psi'\}. \tag{29}$$

The WKB ansatz

$$\psi' = a(X, Y, T)\exp\{i\mu^{-1}\theta(X, Y, T)\},$$

‡Although, this distinction between **M** and S disappears at the level of quasi-geostrophic dynamics.

leads to

$$O(\mu^0): \hat{\omega} = h(\bar{q}_x l - \bar{q}_y k)/(k^2 + l^2), \tag{30}$$

$$O(\mu^1): (\partial/\partial t + \bar{\mathbf{v}} \cdot \nabla_2)(K^2 a) - \hat{\omega}(a\nabla_2 \cdot \mathbf{K} - a\mathbf{K} \cdot \nabla_2 \ln h + 2\mathbf{K} \cdot \nabla_2 a)$$
$$+ (S/h - 2\bar{\mathbf{v}} \cdot \nabla_2 \ln h)(aK^2) + h\bar{q}_x a_Y - h\bar{q}_y a_X = 0, \tag{31}$$

where it has been assumed that the depth h varies on the same scale as the mean flow and

$$\mathbf{K} = (k, l).$$

After considerable algebra (31) can be transformed into an energy equation (see Appendix A)

$$\partial E/\partial t + \nabla_2 \cdot (\mathbf{C}E) = 2EK^{-2}K_i K_j \{\bar{v}_{i,j} - \tfrac{1}{2}\delta_{ij}\nabla_2 \cdot \bar{\mathbf{v}}\} + E\bar{\mathbf{v}} \cdot \nabla_2 \ln h, \tag{32}$$

where

$$E = \tfrac{1}{2}h\overline{\mathbf{v}'^2} = \tfrac{1}{4}h^{-1}a^2K^2.$$

The right-hand side of (32) is the conversion of mean flow kinetic energy to wave energy by Reynolds stresses.

Eq. (25) can be rewritten using standard ray tracing results in two ways. Firstly using

$$dK_i/dt = -K_j\bar{v}_{j,i} - \hat{\omega}_{,i},$$

(see Lighthill, 1978) one obtains

$$\partial P/\partial t + \nabla_2 \cdot (\mathbf{C}P) = \tfrac{1}{2}h^{-1}a^2\{l\mathbf{K} \cdot \nabla_2(h\bar{q}_x) - k\mathbf{K} \cdot \nabla_2(h\bar{q}_y)\} - h\nabla_2 \cdot (h^{-1}\bar{\mathbf{v}})P, \tag{33}$$

where

$$P = (k^2 + l^2)E = \tfrac{1}{2}h\overline{\zeta'^2} = \tfrac{1}{2}h^3\overline{q'^2}. \tag{34}$$

Secondly using

$$d\hat{\omega}/dt = -\hat{C}_i K_j \bar{v}_{i,j} + \bar{v}\hat{\omega}_{,i},$$

(Lighthill, 1978) one has

$$\partial A/\partial t + \nabla_2 \cdot (\mathbf{C}A) = -A(kF_y - lF_x)(k\bar{q}_y - l\bar{q}_x)^{-1}, \tag{35}$$

where

$$A = \hat{\omega}^{-1}E,$$

$$F = \bar{u}\bar{q}_x + \bar{v}\bar{q}_y. \tag{36}$$

P in (33) is the vertically integrated *relative* enstrophy in contrast to the integrated *potential* enstrophy appearing in (11). The right-hand side of (33) simplifies in two circumstances. If h is constant (33) becomes

$$\partial P/\partial t + \nabla_2 \cdot (\mathbf{C}P) = -\nabla_2 \cdot (\bar{\mathbf{v}})P, \tag{37}$$

while if $\beta = 0$ and $h = h_0 \exp(-\alpha x - \beta y)$ then

$$\partial P/\partial t + \nabla_2 \cdot (\mathbf{C}P) = -h\nabla_2 \cdot (h^{-1}\bar{\mathbf{v}})P. \tag{38}$$

In both cases the production of P is related to the horizontal divergence of the mean flow; simple scale analysis gives:

fractional rate of change of $P \sim \nabla \cdot \bar{\mathbf{v}} \sim |\bar{\mathbf{v}}|$ $\{\text{Radius of the Earth}\}^{-1}$,

where it is assumed that the mean flow is in Sverdrup balance. In contrast:

fractional rate of change of $E \sim |\bar{\mathbf{v}}|\bar{L}^{-1}$;

provided $\bar{L}$ is much less than the radius of the Earth P is more nearly conserved than E. The integral of (38) over a large region containing the disturbance is

$$\frac{\partial}{\partial t}\int P\,dA + \int hP\nabla_2 \cdot (h^{-1}\bar{\mathbf{v}})\,dA = 0. \tag{39}$$

This result is derived directly from (22) in Appendix C.

A in (35) is the wave action; A is conserved provided $F = 0$, i.e. if the mean flow is unforced. The general source term in the Rossby wave action equation has not been given before and so the right-hand side of (35) is one of the principal results of this note.

5. DISCUSSION

The wave quantities P and A have different and complementary governing equations [(compare (33) and (35)]. Roughly speaking, the source term in (33) is nonzero when the $\bar{q}$ contours are complicated; in certain cases, such as a constant depth ocean, this source term vanishes and P is conserved. With extremely rough topography, not amenable to WKBJ analysis, topographic scattering produces wave enstrophy very efficiently. The production of A in (35) on the other hand is simply related to $F = \bar{u}\bar{q}_x + \bar{v}\bar{q}_y$.

The slow variation in amplitude of Rossby wave trains is determined at second order in the WKB expansion. At this level of approximation the β-effect is not equivalent to a sloping bottom and a mechanical stress M is not equivalent to Ekman divergence S. It is gratifying that A is conserved in this general case when the mean flow is unforced, this is further evidence for the faithfulness and consistency of the β-plane approximation.

Another major result of this note is embodied in (11); in vertically propagating Rossby waves the enstrophy is conserved even when the mean flow depends on x.

The above analysis does not apply to baroclinic Rossby waves with modal vertical structure (Rhines, 1970); McWilliams (1976) has shown that action is conserved in a two layer fluid when the mean flow is unforced. The full baroclinic problem is complicated by rapid vertical variation and nonseparability (similar to that which occurs in the familiar baroclinic instability problem).

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Appendix A

DERIVATION OF THE ENERGY EQUATION

In this appendix (9) and (32) are derived; we prefer to obtain these energy equations from the equations of motion; they also follow from the WKB transport equations (8) and (31).

To get (9), start with the linearized, geostrophic Boussinesq equations of motion, retaining order Rossby number terms,

$$\bar{D}\mathbf{v}' + (\mathbf{v}' \cdot \nabla_2)\bar{\mathbf{v}} - \hat{\mathbf{z}} \times f\mathbf{v}' + f_0 \nabla_2 \psi' = O(\text{Rossby number})^2, \tag{A1}$$

$$\bar{D}\partial\psi'/\partial z + (\mathbf{v}' \cdot \nabla_2)\partial\bar{\psi}/\partial z + w' N^2 f_0^{-1} = O(\text{Rossby number})^2, \tag{A2}$$

$$\nabla_2 \cdot \mathbf{v}' + w'_z = O(\text{Rossby number})^2, \tag{A3}$$

where

$$\mathbf{v}' = (u', v'), \tag{A4}$$

$$\bar{D} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}. \tag{A5}$$

Forming the combination $\overline{\mathbf{v}' \cdot (\mathrm{A1})} + N^{-2}\overline{(\mathrm{A3})\partial\psi'/\partial z}$ one has

$$\bar{D}[\tfrac{1}{2}\overline{\mathbf{v}' \cdot \mathbf{v}'} + \tfrac{1}{2}N^{-2}\overline{(\partial\psi'/\partial z)^2}] + \nabla \cdot \{\overline{\psi'(\mathbf{v}' + w'\hat{\mathbf{z}})}\}$$

$$= -\overline{v_i' v_j'}\,\bar{v}_{i,j} + f_0 N^{-2}\overline{(\partial\psi'/\partial z) v_i'}\,(\partial\bar{\psi}/\partial z)_{,i}. \quad \text{(A6)}$$

Using the WKB ansatz (6) to evaluate the terms in (A6) to lowest nonzero order one recovers (9).

To get (32) start with the shallow water equations

$$\bar{D}\mathbf{v}' + (\mathbf{v}' \cdot \nabla_2)\bar{\mathbf{v}} - \hat{z} \times f\mathbf{v}' + f_0 \nabla p' = 0, \quad \text{(A7)}$$

$$\nabla_2 \cdot (h\mathbf{v}') = 0. \quad \text{(A8)}$$

Taking $h\mathbf{v}' \cdot$(A7) one has

$$(\partial/\partial t)(\tfrac{1}{2}h\overline{\mathbf{v}' \cdot \mathbf{v}'}) + \nabla_2 \cdot (\tfrac{1}{2}h\overline{\mathbf{v}' \cdot \mathbf{v}'}\,\bar{\mathbf{v}} + f_0 h\overline{p'\mathbf{v}'}) = -h\overline{v_i' v_j'}\,\bar{\mathbf{v}}_{i,j} + S\tfrac{1}{2}\overline{\mathbf{v}' \cdot \mathbf{v}'}, \quad \text{(A9)}$$

where [see (26) and the subsequent discussion]

$$S = \nabla_2 \cdot (h\bar{\mathbf{v}}). \quad \text{(A10)}$$

Using the WKB ansatz together with (A10) recovers (32) from (A9).

Appendix B

DERIVATION OF THE BAROTROPIC POTENTIAL VORTICITY EQUATION

The shallow water equations in an ocean of varying depth h are

$$\partial\mathbf{v}/\partial t + \hat{\mathbf{z}} \times hq\mathbf{v} = -\nabla B + \mathbf{M}, \quad \text{(B1)}$$

$$\nabla_2 \cdot h\mathbf{v} = S, \quad \text{(B2)}$$

where

$B = \rho^{-1}p + \tfrac{1}{2}\mathbf{v} \cdot \mathbf{v}$,

$\mathbf{M}$ = mechanical stresses,

S = mass source term,

$q = (\zeta + f)/h$ = barotropic potential vorticity.

There are two forcing mechanisms, $\mathbf{M}$ and S. The mass source S is a more realistic method of representing the divergent upper Ekman layer than the mechanical stress **M**.

To obtain the barotropic potential vorticity equation take the curl of (B1) and use (B2) to get

$$\partial q/\partial t + \mathbf{v}\cdot\nabla_2 q = h^{-1}\{\nabla\times\mathbf{M}\cdot\hat{\mathbf{z}} - qS\}. \tag{B3}$$

The linearized fluctuation equation (22) follows from (B3). Note that if **M** and S are mean forcing terms, S appears in the fluctuation equation but **M** does not.

Appendix C

DERIVATION OF THE INTEGRATED RELATIVE ENSTROPHY EQUATION (33)

Consider an ocean with $\beta=0$ and $h=h_0\exp(-\gamma y)$, so the potential vorticity equation (22) can be put in the form

$$(\partial/\partial t + \bar{\mathbf{v}}\cdot\nabla_2)(h^2 q') + (h^{-1}S - 2\bar{\mathbf{v}}\cdot\nabla_2 \ln h)(h^2 q') + \gamma f_0 h v' = 0, \tag{C1}$$

$$h^2 q' = \nabla_2^2\psi' + \gamma\psi'_y. \tag{C2}$$

If (C1) is multiplied by hq', integrated over a large area and averaged (33) is recovered. In particular then the third term in (C1) vanishes completely since

$$\int\int \exp(\gamma y)\,\psi'_x(\psi'_{xx}+\psi'_{yy}+\gamma\psi'_y)\,dx\,dy$$

$$= \int\int \psi'_x \frac{\partial}{\partial y}(\exp(\gamma y)\,\psi'_y)\,dx\,dy$$

$$= -\int \exp(\gamma y)\left\{\int \psi'_{xy}\psi'_y\,dx\right\}dy = 0.$$

Note that if (C1) is multiplied by $h^n q'$ the third term will vanish only when $n=1$; this suggests that out of the family of wave properties $h^m q'^2$ the member $m=3$ will have the simplest conservation properties.

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