The Subinertial Mixed Layer Approximation

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ABSTRACT

The density of the mixed layer is approximately uniform in the vertical but has dynamically important horizontal gradients. These nonuniformities in density result in a vertically sheared horizontal pressure gradient. Subinertial motions balance this pressure gradient with a vertically sheared velocity. Systematic incorporation of shear into a three-dimensional mixed layer model is both the goal of the present study and its major novelty.

The sheared flow is partitioned between a geostrophic response and a frictional, ageostrophic response. The relative weighting of these two components is determined by a nondimensional parameter $\mu \equiv 1/f\tau_U$, where τ_U is the timescale for vertical mixing of momentum and f^{-1} is the inertial timescale.

If μ is of order unity, then the velocity has vertical shear at leading order. Differential advection by this shear flow will tilt over vertical isosurfaces of heat and salt so as to "unmix" or "restratify" the mixed layer. The unmixing process is balanced by intermittent mixing events, which drive the mixed layer back to a state of vertical homogeneity.

All of these processes are captured by a new set of reduced or filtered dynamics called the subinertial mixed layer (SML) approximation. The SML approximation is obtained by expanding the equations of motion in both Rossby number and a second small parameter that is the ratio of the vertical mixing timescale to the dynamic time scale. The subinertial dynamics of slab mixed layer models is captured as a special case of the SML approximation by taking the limit $\mu \to \infty$.

1. Introduction

The goal of this paper is to develop a set of approximate dynamics describing the subinertial motions in a vertically homogeneous and rapidly rotating layer of fluid with a free lower boundary. This is a very idealized model of the ocean mixed layer (ML). The simplified equations of motion, called the subinertial mixed layer (SML) approximation, are obtained by applying standard "filtering" arguments to the primitive equations. The analysis, and the resulting approximate dynamics, is similar to the quasigeostrophic set (Pedlosky 1987).

The ML is the uppermost layer of the ocean in which recent mixing events have created almost vertically homogeneous temperature and salinity fields. Many previous models of the ML have also assumed that the velocity is vertically uniform or "slablike." Examples of slab models include Schopf and Cane (1983), de Szoeke and Richman (1984), McCreary and Kundu (1988), and McCreary and Yu (1992). A recent discussion of the conservation laws and stability properties of slab models is given by Ripa (1993). One important issue confronted in the course of deriving the SML approximation is the basis for the assumption of a slab velocity profile in the ML.

At first sight, a slab velocity profile seems to be demanded by the vertical homogeneity of the temperature and salinity in the ML: otherwise the differential advection due to the sheared velocity profile would tilt over vertically homogeneous property surfaces and "restratify" the ML. In fact, the role of shear-driven restratification is a central issue of this paper and also in recent work by Tandon and Garrett (1994) and Roemmich et al. (1994). Along with these authors, I argue that there are compelling reasons for believing that there are sheared velocity fields within the ML, and that this shear does result in a partial restratification of the ML. Opposing the restratification process are intermittent mixing events, such as storms or shear flow instabilities.

The balance between differential advection and vertical mixing outlined above is the same as that in Taylor's (1953) theory of shear dispersion. In the ML the mixing is most probably episodic, rather than the continually acting molecular diffusion in Taylor's theory. But the essential idea of *approximate* vertical uniformity, corrected by a vertical structure reflecting the competition between differential advection and vertical mixing, is common to both the present work and shear dispersion. Certainly the example of shear dispersion shows that there is not necessarily a contradiction be-

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tween almost vertically homogeneous tracer fields and advection by a strongly sheared velocity.

But is the velocity in the ML "strongly sheared"? At first sight one might believe that the mixing that homogenizes the temperature and salinity would also ensure that the momentum is uniformly mixed. There are two reasons why this might not be the case.

First, there is an additional timescale in the momentum equations not present in the tracer equations, namely, f^{-1}. If the timescale for the vertical mixing of momentum[1] is τ_U then a slab velocity profile requires that

$$\tau_U \ll f^{-1} \tag{1.1}$$

so that the momentum mixing is faster than the inertial timescale. By contrast, if the timescale for the vertical mixing of tracers is τ, then vertical uniformity of heat and salt requires only that

$$\tau \ll \mathcal{T}. \tag{1.2}$$

In (1.2) $\mathcal{T}$ is the timescale characteristic of the subinertial evolution of the ML so that the Rossby number is

$$\mathrm{Ro} \equiv \frac{1}{f\mathcal{T}} \ll 1. \tag{1.3}$$

If one makes the natural assumption that $\tau \sim \tau_U$, then the inequality in (1.1) is more restrictive than those in (1.2) and (1.3). To summarize, if $\mathcal{T} \gg \tau \sim \tau_U \gg f^{-1}$ there is the possibility that heat and salt are almost vertically uniform whereas the velocity is not. The detailed derivation of the SML approximation in section 3 "fleshes out" this scale analysis.

There is a second reason to anticipate that the ML velocity might be sheared, while the heat and salt are vertically uniform. Using the hydrostatic relation, the pressure in the mixed layer is

$$p = \rho g(z + h) + p(-h), \tag{1.4}$$

where $\rho(x, y, t)$ is the horizontally inhomogeneous density and $h(x, y, t)$ is the depth of the ML. The constant of integration $p(-h)$ is the pressure at the base of the ML. It immediately follows from (1.4) that there is a systematic and persistent depth-dependent forcing ∇p in the horizontal momentum equations. There is no analog of this depth-dependent force in the conservation equations for heat and salt. Slab mixed layer models either ignore the depth-dependent pressure gradient, or assume that Reynolds stresses are such as to precisely cancel this inconvenient term.

In addition to the theoretical arguments outlined above, there is the possibility of observing either sheared currents or shear-driven restratification within the ML. As an example of the latter possibility, Roemmich et al. (1994) use XCTD observations of salinity in the equatorial ML to infer that salinity restratification occurs when the westward wind stress relaxes. This equatorial restratification is created by a vertically sheared zonal current accelerating in response to a vertically sheared zonal pressure gradient. The shears are substantial: Roemmich et al. estimate surface velocities of about 1 m s^{-1} eastward and a shear of 0.4 m s^{-1} in the top 100 m of the equatorial Pacific.

Using Lagrangian drifter data from the northeast Pacific, Niiler and Paduan (1994) infer the existence of sheared currents in the ML. These authors attribute the shear to a turbulent stress profile with significant curvature. Because of this curvature with depth, the divergence of the stress is a body force with vertical structure, rather than the vertically uniform body force assumed by slab models. This mechanism is different from the depth-dependent pressure force mentioned above, but the upshot is the same: sheared ML currents are driven by a nonslab body force.

A third observational study of ML shear is the MILE experiment described by Davis et al. (1981). During the 20-day course of this experiment, closely spaced VACMs measured the velocity in the top 50 m of the ocean. The results, summarized in Fig. 17 of Davis et al., show significant shears in the ML except during a strong wind event. This event lasted two days and resulted in a reduction of ML shear. A second weaker and less persistent wind event had no noticeable effect on ML shear. The conclusion of Davis et al. is that the MILE experiment did not confirm the slab ML model except during strong wind events.

The meteorological slab mixed layer model was formulated by Lavoie (1972). A recent paper reviewing the meteorological literature is Dempsey and Rotunno (1988). These authors use a slab mixed layer to model topographic generation of mesoscale vortices such as the Denver cyclone. The larger database in the atmospheric sciences provides a better observational test of the slab mixed layer model than is available in oceanography. The conclusion of Dempsey and Rotunno is that observations do not support the assumption that the velocity is well mixed in the atmospheric planetary boundary layer.

In the next section, I introduce the primitive equations of mixed layer dynamics. Apart from the model adopted for the parameterization of vertical mixing, this formulation is standard. In section 3 the SML equation is derived using an asymptotic expansion whose principal parametric assumptions are the inequalities (1.2) and (1.3). The assumption (1.1) is not required in the derivation of the SML approximation. But the limit in (1.1) can be taken a posteriori within

[1] For example, if one models the mixing with a vertical eddy viscosity ν and the order of magnitude of the ML depth is $\mathcal{H}$, then $\tau_U \sim \mathcal{H}^2/\nu$. Likewise, if the tracer mixing is modeled with a vertical diffusivity κ, then $\tau \sim \mathcal{H}^2/\kappa$.

the context of the SML approximation so that the subinertial dynamics of the slab models is captured as a special case.

2. Formulation of a tractable model

a. *The equations of motion*

In this section I formulate a model of mixed layer dynamics. The mixed layer is the domain $0 > z > -h(x, y, t)$ and the model dynamics in this region are

$$\frac{Du}{Dt} - fv + p_x = \tau_U^{-1}(\bar{u} - u) + h^{-1}\mathcal{F}_u$$

$$\frac{Dv}{Dt} + fu + p_y = \tau_U^{-1}(\bar{v} - v) + h^{-1}\mathcal{F}_v$$

$$p_z - b = 0$$

$$u_x + v_y + w_z = 0$$

$$\frac{DT}{Dt} = \tau^{-1}(\bar{T} - T) + h^{-1}\mathcal{F}_T$$

$$\frac{DS}{Dt} = \tau^{-1}(\bar{S} - S) + h^{-1}\mathcal{F}_S, \quad (2.1\text{a–f})$$

where $D/Dt \equiv \partial_t + u\partial_x + v\partial_y + w\partial_z$ is the convective derivative. The equations above are not yet vertically averaged: all of the variables depend on x, y, z, and t.

In (2.1c) b denotes the buoyancy so that the density is

$$\rho = \check{\rho}(1 - g^{-1}b), \quad b = g[\alpha_T(T - \check{T}) - \alpha_S(S - \check{S})], \quad (2.2\text{a,b})$$

where $\check{\rho}$, $\check{T}$, and $\check{S}$ are the constant density, temperature, and salinity immediately below the surface $z = -h$. For static stability at the base of the mixed layer $b(x, y, -h, t) > 0$. The total pressure in the mixed layer is

$$P = -\check{\rho}gz + \check{\rho}p, \quad (2.3)$$

where p is related to b by (2.1c). Some of the notation is summarized in Fig. 1.

The sources $\mathcal{F}_u(x, y, z, t)$, $\mathcal{F}_v(x, y, z, t)$, $\mathcal{F}_S(x, y, z, t)$, and $\mathcal{F}_T(x, y, z, t)$ in the conservation equations represent the input of momentum, heat, and salt from either the atmosphere above or the ocean interior below.

In (2.1) the overbar denotes the vertical average of a field:

$$\bar{\theta}(x, y, t) \equiv \frac{1}{h}\int_{-h}^{0} \theta(x, y, z, t)dz,$$

$$\theta(x, y, z, t) = \bar{\theta}(x, y, t) + \theta'(x, y, z, t). \quad (2.4\text{a,b})$$

I also use the notation:

$$\hat{\theta}(x, y, t) \equiv \theta(x, y, -h^+, t),$$

$$\check{\theta}(x, y, t) \equiv \theta(x, y, -h^-, t). \quad (2.5\text{a,b})$$

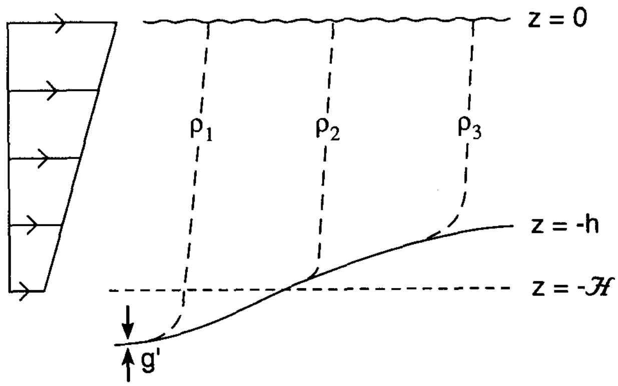

FIG. 1. A definition sketch of the idealized mixed layer. A rapidly rotating, buoyant layer of fluid floats on a lower layer. There is reduced gravity, g', at the interface between the two layers. The upper layer is vertically homogenous in temperature, salinity, and density, but these three properties do vary horizontally. The horizontal pressure gradient in the upper layer has a component due to the vertical displacement of the interface and a second component due to the horizontal variations in density within the layer. If the horizontal pressure gradient is balanced geostrophically, then the resulting velocity is vertically sheared as indicated at left. The shear will tilt over the property surfaces and produce vertical gradients in temperature, salinity, and density. This shear-driven "unmixing" is checked by vertical mixing so that the vertical homogeneity of the layer is maintained.

Thus $\hat{\theta}$ is the value of θ just above the base of the mixed layer and $\check{\theta}$ is the value just below the base of the mixed layer. Some fields, such as pressure, are continuous so that $\hat{p} = \check{p}$. Others, such as temperature, are effectively discontinuous. Thus, the temperature jump at the base of the mixed layer is $\hat{T} - \check{T}$.

The base of the mixed layer is at $z = -h(x, y, t)$ and since this is not always a material surface its motion is described using the well-known concept of an entrainment velocity:

$$h_t + \hat{u}h_x + \hat{v}h_y + \hat{w} = w_{\text{en}} \quad (2.6)$$

(e.g., de Szoeke 1980; Schopf and Cane 1983; de Szoeke and Richman 1984). If the entrainment velocity, $w_{\text{en}}(x, y, t)$, is positive, then fluid is passing through the surface $z = -h(x, y, t)$ and entering the mixed layer.

b. *Vertical averages of the equations of motion*

A useful identity is

$$\overline{\frac{D\theta}{Dt}} = \frac{1}{h}[\partial_t(h\bar{\theta}) + \nabla\cdot(h\overline{\mathbf{u}\theta}) - w_{\text{en}}\hat{\theta}], \quad (2.7)$$

where $\nabla \equiv (\partial_x, \partial_y)$. The result (2.7) is proved with a sedulous application of Leibnitz's rule.

The vertical average of the conservation equations in (2.1) can be taken using (2.7). For example, considering S in (2.1f) and using (2.7) gives

$$(h\bar{S})_t + \nabla\cdot[h\bar{\mathbf{u}}\bar{S} + h\overline{\mathbf{u}'S'}] = w_{\text{en}}\hat{S} + \bar{\mathcal{F}}_S. \quad (2.8)$$

Physical considerations now show that one must have

$$\bar{\mathcal{F}}_S = \mathcal{Q}_S + H(w_{en})w_{en}(\check{S} - \hat{S}), \qquad (2.9)$$

where $\mathcal{Q}_S(x, y, t)$ is the effective source of salinity due to differences between evaporation and precipitation at the surface and $H(x)$ is the Heaviside step function. The other final term on the right-hand side of (2.9) is included so that the total conservation law in (2.8) now becomes

$$(h\bar{S})_t + \nabla \cdot [h\bar{\mathbf{u}}\bar{S} + h\overline{\mathbf{u}'S'}]$$
$$= \mathcal{Q}_S + H(w_{en})w_{en}\check{S} + H(-w_{en})w_{en}\hat{S}. \qquad (2.10)$$

If $w_{en} > 0$, then fluid is entering the mixed layer and the second term on the rhs of (2.10) is the input of salinity from the ocean interior. If $w_{en} < 0$, then fluid is leaving the layer and the last term on the right-hand side of (2.10) is the loss of salinity accompanying this detrainment.

The vertically averaged conservation equations for temperature and momentum can be constructed in an analogous way using the identity (2.7). In each case we decompose the vertically averaged source $\bar{\mathcal{F}}_{\mathcal{X}}$, where $\mathcal{X} = u, v, S$, or T as

$$\bar{\mathcal{F}}_{\mathcal{X}} = \mathcal{Q}_{\mathcal{X}} + H(w_{en})w_{en}(\check{\mathcal{X}} - \hat{\mathcal{X}}). \qquad (2.11)$$

Here $\mathcal{Q}_{\mathcal{X}}(x, y, t)$ is the source at the surface or lateral diffusion. The decomposition in (2.11) ensures that in the total conservation law, analogous to (2.10), the source terms introduced by the entrainment w_{en} are correct. Equation (2.11) is a physical constraint that any model of $\mathcal{F}_{\mathcal{X}}$ and w_{en} used in conjunction with "relaxation to vertical average" must satisfy.

The appendix A contains a derivation of the vertically averaged, vertical vorticity balance of the system in (2.1). The most important result in that appendix is the conservation law in (A.11), which is used in section 3 to assist in the derivation of the SML approximation.

c. A summary of the forcing functions

The mixed layer is forced by the fluxes $\mathcal{F}_{\mathcal{X}}$, by entrainment w_{en}, and finally by pressure forces at $z = -h$. This means that in order to consider (2.1a–f) and (2.2) as a closed set of equations for the seven unknown independent variables u, v, w, p, b, S, and T we must prescribe or parameterize the functions $\mathcal{F}_{\mathcal{X}}$, w_{en}, $\check{\mathcal{X}}$, and also $\hat{p} = \check{p}$. This final external function is the constant of integration needed to determine p from (2.1c).

We do not attempt to model the dynamics of the region below the mixed layer and so we simply prescribe $\hat{p}$ as some given forcing function:

$$\hat{p} = f\varphi(x, y, t). \qquad (2.12)$$

Because of the assumption that the fluid in the layer below $z = -h$ has uniform density, $\check{\rho}$, the total pressure in this layer is $P = -gz\check{\rho} + \check{\rho}f\varphi$. Thus the special case $\varphi = 0$ corresponds to the 1½-layer model in which it is assumed that the pressure gradient in the lower layer is zero. Equation (2.12) completes the description of the mixed layer model.

d. A discussion of the mixing parameterization

The most unfamiliar terms in (2.1) are those introduced by the parameterization of vertical mixing. The model in (2.1) uses "relaxation to vertical average" with a time constant τ_U in the momentum equations and τ in the heat and salt equations. In defense of this model assumption I argue that

(i) it is less ad hoc than simply discarding some of the depth-dependent terms in the equations of motion;

(ii) for the length and timescales of interest relaxation to the vertical average is not more ad hoc than vertical eddy diffusivities and viscosities;

(iii) the ultimate goal is a set of equations describing the nonlinear evolution of the subinertial dynamics of the system in (2.1). The form of these equations does not depend on the details of how vertical mixing is parameterized: vertical eddy diffusivity leads to the same result.

Notice that the "depth of the mixed layer" is defined as the depth over which the relaxation to vertical average operates: $h(x, y, t)$ enters through the definition of vertical average in (2.4). It is possible to refine the vertical relaxation parameterization by making the relaxation time τ depend on the other model variables. An obvious suggestion is $\tau \sim h/u_*$, where u_* is the friction velocity that parameterizes the generation of turbulent energy (u_*^3). I do not follow this route in the present article, though the asymptotic development in section 3 easily accomodates such complications.

The vertical homogeneity of the mixed layer is maintained by events such as storms, or perhaps intermittent overturns. The parameters τ and τ_U are roughly the *average time* between these events (i.e., several days) and not the *duration* of the events (i.e., a few hours).

Between mixing events, the layer is actually unmixing, that is, developing vertical structure or "restratifying." One unmixing mechanism is the differential advection by the shear flow on the left-hand side of Fig. 1: vertical isosurfaces will be tilted over so that vertical gradients are created from horizontal gradients (e.g., Roemmich et al. 1994; Tandon and Garrett 1994). A second unmixing mechanism is the vertical structure of the forcing functions $\mathcal{F}_u$, $\mathcal{F}_v$, $\mathcal{F}_S$, and $\mathcal{F}_T$. It is unlikely that atmospheric forcing is distributed uniformly over the mixed layer, so the selective action of atmospheric inputs on the top of the layer will create vertical structure.

It might be objected that, if the shear flow is geostrophically balanced, then it is parallel to density surfaces and so does not affect density. But not all of the vertically sheared pressure gradient will be balanced geostophically: in the mixed layer there is some vertical mixing of momentum [modeled by τ_U in (2.1a,b)] and this friction leads to a component of flow down the pressure gradient. The resulting vertically sheared velocity will tilt the density surfaces. A second consideration is that density is determined by both heat and salt, and provided that these tracers are independent their separate isosurfaces will be tilted even by a geostrophically balanced shear flow.

3. The SML expansion

For brevity the effects of forcing are ignored in this first derivation of the SML approximation: from this point forward take $\mathcal{F}_{\mathcal{X}} = 0$ and $w_{en} = 0$. Provided that the forcing is sufficiently gentle so as not to disrupt the leading order balances its inclusion within the SML approximation is straightforward. But it is instructive to retain the pressure forcing function φ in (2.12).

a. The basic state and the length and timescales

The expansion is a perturbation about a basic state in which the mixed layer has a uniform depth $\mathcal{H}$, uniform temperature $\langle \bar{T} \rangle$, and uniform salinity $\langle \bar{S} \rangle$. The fluid below the base of the mixed layer also has uniform temperature $\check{T}$ and salinity $\check{S}$. Thus, there is a reduced gravity

$$g' \equiv g[\alpha_T(\langle \bar{T} \rangle - \check{T}) - \alpha_S(\langle \bar{S} \rangle - \check{S})], \quad (3.1)$$

where the angle brackets in (3.1) denote an integral average of a horizontal area: $\langle S \rangle \equiv A^{-1} \int S dA$, where $dA = dxdy$. For instance, the basic-state depth of the mixed layer is defined as $\mathcal{H} \equiv \langle h \rangle$. The Rossby radius of deformation is

$$\mathcal{L}^2 \equiv \frac{g'\mathcal{H}}{f^2}. \quad (3.2)$$

A typical reduced gravity at the mixed layer base is $g' \sim 10^{-2}\ \mathrm{m^2\ s^{-1}}$, so that if $\mathcal{H} \sim 100$ m and $f \sim 10^{-4}\ \mathrm{s^{-1}}$ then $\mathcal{L} \sim 10$ km.

The problem contains four timescales. The first three timescales are apparent in (2.1): they are the inertial timescale f^{-1} and the two mixing times τ and τ_U. The fourth timescale is the "dynamic timescale" denoted by $\mathcal{T}$. As in the derivation of the quasigeostrophic approximation, the dynamic timescale is characteristic of the subinertial evolution of the vertical vorticity. Thus the scale of the horizontal velocity is

$$\mathcal{U} \equiv \frac{\mathcal{L}}{\mathcal{T}}, \quad (3.3)$$

and the Rossby number is

$$\mathrm{Ro} \equiv \frac{1}{f\mathcal{T}} = \frac{\mathcal{U}}{f\mathcal{L}}. \quad (3.4)$$

The expansion requires that $\mathrm{Ro} \ll 1$.

As an explicit definition of $\mathcal{T}$ one might use

$$\mathcal{T} \equiv \frac{f\mathcal{L}}{\Gamma\mathcal{H}}, \quad (3.5)$$

where Γ is the scale of the *horizontal* buoyancy gradient within the mixed layer, for example, $\Gamma^2 \equiv \langle \nabla B \cdot \nabla B \rangle$. The timescale in (3.5) follows from scale analysis of a thermal wind balance $f\mathcal{U}/\mathcal{H} \sim \Gamma$. To estimate an order of magnitude for $\mathcal{T}$, assume a large-scale horizontal buoyancy gradient produced by a temperature change of 3°C in 1000 km. This thermal gradient gives

$$\Gamma \sim g\alpha_T \frac{dT}{dy} \sim 10^{-8}\ \mathrm{s^{-2}}. \quad (3.6)$$

Using the numbers above for $\mathcal{L}$, etc., one finds that $\mathcal{T} \sim 10^6$ s. I emphasize that this estimate of the dynamic timescale is based on a climatological temperature gradient: in frontal regions Γ is at least a factor of 10 larger than the estimate in (3.6) (e.g., Samelson and Paulson 1988) and the dynamic timescale $\mathcal{T}$ is correspondingly reduced.

From the four timescales τ, τ_U, f^{-1}, and $\mathcal{T}$ one can form three nondimensional numbers. The first is the Rossby number introduced in (3.4). The other two nondimensional numbers are now defined as

$$\epsilon \equiv \frac{\tau}{\mathcal{T}} \quad (3.7)$$

and

$$\mu \equiv \frac{1}{f\tau_U}. \quad (3.8)$$

The expansion requires that both Ro and ϵ be small but no restriction is placed on μ. We pursue a general development by carrying μ as order unity. Although this complicates the algebra, it has the advantage of establishing contact with the slab mixed layer models discussed in the introduction. The subinertial dynamics of the slab models can be recovered as a special case of the SML approximation by taking the limit $\mu \to \infty$.

b. Nondimensional variables

The expansion is best done in dimensionless variables, denoted by an asterisk:

$$(x, y) = \mathcal{L}(x_*, y_*), \quad z = \mathcal{H}z_*, \quad t = \mathcal{T}t_*,$$
$$(u, v) = \mathcal{U}(u_*, v_*), \quad w = (\mathcal{H}/\mathcal{L})\mathcal{U}w_*, \quad (3.9)$$

where $\mathcal{U}$ is defined in (3.3).

The layer depth is written as

$$h = \mathscr{H}(1 + \mathrm{Ro}\eta_*), \qquad (3.10)$$

where $\mathscr{H}$ is the undisturbed, constant depth of the layer. If η_* is of order one, then there are small [O(Ro)] changes in the layer depth. This is the usual quasigeostrophic scaling (Pedlosky 1987). Notice that in dimensionless variables the bottom of the layer is at $z_* = -1 - \mathrm{Ro}\eta_*$ and because of our definition of $\mathscr{H}$, $\langle \eta_* \rangle = 0$.

The total temperature and salinity in the mixed layer is written as

$$T = \langle \bar{T} \rangle + \mathrm{Ro}(g'/g\alpha_T)T_*,$$
$$S = \langle \bar{S} \rangle + \mathrm{Ro}(g'/g\alpha_S)S_*. \qquad (3.11)$$

Here $T_*(x, y, z, t)$ and $S_*(x, y, z, t)$ are the nondimensional perturbations of temperature and salinity superimposed on the uniform basic-state values $\langle \bar{T} \rangle$ and $\langle \bar{S} \rangle$. Introducing the definition of g' in (3.1) into (2.2b) gives

$$b = g'(1 + \mathrm{Ro}b_*), \quad \text{where} \quad b_* \equiv T_* - S_*. \qquad (3.12\text{a,b})$$

The nondimensional buoyancy perturbation is b_*, and clearly $\langle \bar{T}_* \rangle = \langle \bar{S}_* \rangle = \langle \bar{b}_* \rangle = 0$. The nondimensional units have been constructed so that if $T_* = S_*$, then the density effects of temperature and salinity perfectly cancel; then the buoyancy of the layer is equal to g' defined in (3.1).

The pressure is

$$p = g'(z + \mathscr{H}) + f\mathscr{U}\mathscr{L}p_*, \quad \varphi = \mathscr{U}\mathscr{L}\varphi_*. \quad (3.13\text{a,b})$$

Using these definitions the nondimensional version of the lower boundary condition in (2.14) is $p_*(x_*, y_*, -1 - \mathrm{Ro}\eta_*, t_*) = \eta_* + \varphi_*$.

Introducing all of the definitions above into the dimensional equations of motion and then dropping the $*$'s gives the nondimensional set

$$\mathrm{Ro}\frac{Du}{Dt} - v + p_x = -\mu u'$$
$$\mathrm{Ro}\frac{Dv}{Dt} + u + p_y = -\mu v'$$
$$p_z - b = 0$$
$$u_x + v_y + w_z = 0$$
$$\epsilon\frac{DS}{Dt} = -S'$$
$$\epsilon\frac{DT}{Dt} = -T'$$
$$b = T - S. \qquad (3.14\text{a–g})$$

The nondimensional version of the kinematic boundary condition in (2.6) is

$$\mathrm{Ro}(\eta_t + \hat{u}\eta_x + \hat{v}\eta_y) + \hat{w} = 0, \qquad (3.15)$$

and the pressure boundary condition in (2.12) is

$$p(x, y, -1 - \mathrm{Ro}\eta, t) = \eta + \varphi. \qquad (3.16)$$

The nondimensional problem is completely defined by (3.14), (3.15), and (3.16).

c. The expansion

The asymptotic expansion assumes that both ϵ and Ro are small. More precisely, we take a distinguished limit in which $\epsilon \to 0$ and

$$\mathrm{Ro} = \epsilon^2\mathscr{R}, \qquad (3.17)$$

where $\mathscr{R}$ is of order unity.

All of the independent variables are expanded in powers of ϵ:

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots. \qquad (3.18)$$

Our goal is to capture all terms of order ϵ^3. This level of accuracy requires that we calculate higher-order corrections to the leading-order evolution equations; so a multiple timescale expansion is needed:

$$\partial_t = \partial_{t_0} + \epsilon\partial_{t_1} + \cdots. \qquad (3.19)$$

Incidentally, the QG approximation stops at terms of order ϵ^2 and we penetrate one order higher in ϵ than this. The balance models described by Allen (1991), McWilliams and Gent (1980), and Salmon (1985) are generally concerned with terms of order $\mathrm{Ro}^2 \sim \epsilon^4$ and this, in turn, is one order higher in ϵ than the SML approximation. However, the balance models deal with homogeneous layers and in that case the expansion is in integer powers of $\mathrm{Ro} \sim \epsilon^2$, while in the SML approximation we must consider all powers of ϵ.

The boundary conditions at $z = -1 - \epsilon^2\mathscr{R}\eta$ in (3.15) and (3.16) must also be expanded so that they can be applied at $z = -1$. Thus in the kinematic boundary condition (3.15), one has $\hat{w} \equiv w(-1 - \epsilon^2\mathscr{R}\eta) = w(-1) - \epsilon^2\mathscr{R}\eta w_z(-1) + O(\epsilon^4)$. Then $w(-1)$ and $w_z(-1)$ are expanded as in (3.18) and terms of equal order in (3.15a) are collected. The result is

$$w_0(-1) = 0$$
$$w_1(-1) = 0$$
$$\mathscr{R}[\partial_{t_0}\eta_0 + u_0(-1)\eta_{0x} + v_0(-1)\eta_{0y}] + w_2(-1) - \mathscr{R}\eta_0 w_{0z}(-1) = 0. \quad (3.20\text{a–c})$$

Applying this same procedure to the pressure boundary condition in (3.16) one finds

$$p_0(-1) = \eta_0 + \varphi$$
$$p_1(-1) = \eta_1$$
$$p_2(-1) = \eta_2 + \mathcal{R}\eta_0 b_0(-1). \quad (3.21\text{a–c})$$

There is one final complication introduced by the moving boundary at $z = -1 - \epsilon^2\mathcal{R}\eta$. The vertical average of $\theta = \theta_0 + \epsilon\theta_1 + \cdots$ is

$$\begin{aligned}
\bar{\theta} &\equiv (1 + \epsilon^2\mathcal{R}\eta)^{-1}\int_{-1-\epsilon^2\mathcal{R}\eta}^{0} \theta dz \\
&= (1 - \epsilon^2\mathcal{R}\eta)\left[\int_{-1}^{0} \theta dz + \int_{-1-\epsilon^2\mathcal{R}\eta}^{-1} \theta dz\right] + \mathrm{O}(\epsilon^3) \\
&= \int_{-1}^{0} \theta_0 dz + \epsilon\int_{-1}^{0} \theta_1 dz + \epsilon^2\left[\int_{-1}^{0} \theta_2 dz \right.\\
&\quad \left. + \mathcal{R}\eta_0\left(\theta_0(-1) - \int_{-1}^{0} \theta_0 dz\right)\right] + \mathrm{O}(\epsilon^3) \\
&= \tilde{\theta}_0 + \epsilon\tilde{\theta}_1 + \epsilon^2[\tilde{\theta}_2 + \mathcal{R}\eta_0(\theta_0(-1) - \tilde{\theta}_0)] + \mathrm{O}(\epsilon^3),
\end{aligned} \quad (3.22\text{a–d})$$

where in (3.22d) I have introduced the notation $\tilde{\theta} \equiv \int_{-1}^{0} \theta dz$. Notice that in the transition from (3.22b) to (3.22c), I have used the intermediate result $\int_{-1-\epsilon^2\mathcal{R}\eta}^{-1} \theta dz = \epsilon^2\mathcal{R}\eta_0\theta_0(-1) + \mathrm{O}(\epsilon^3)$.

The $\mathrm{O}(\epsilon^2)$ term in (3.22d) shows that it is important to preserve the distinction between an average over the entire layer depth (denoted by $\bar{\theta}$) and an average over the unperturbed layer depth (denoted by $\tilde{\theta}$).

Because of the additional terms at $\mathrm{O}(\epsilon^2)$, one must be careful when expanding θ'. Using (3.18d), one has

$$\begin{aligned}
\theta' &\equiv \theta - \bar{\theta} \\
&= [\theta_0 - \tilde{\theta}_0] + \epsilon[\theta_1 - \tilde{\theta}_1] \\
&\quad + \epsilon^2[\theta_2 - \tilde{\theta}_2 - \mathcal{R}\eta_0(\theta_0(-1) - \tilde{\theta}_0)] \\
&= \theta'_0 + \epsilon\theta'_1 + \epsilon^2\theta'_2 + \mathrm{O}(\epsilon^3),
\end{aligned} \quad (3.23\text{a–c})$$

and so on. Notice that $\theta'_2 \neq \theta_2 - \tilde{\theta}_2$ and $\tilde{\theta}'_2 = -\mathcal{R}\eta_0(\theta_0(-1) - \tilde{\theta}_0)$.

d. The terms of $O(\epsilon^0)$

Collecting the terms of order ϵ^0 in the equations of motion (3.14) one has

$$\begin{aligned}
-v_0 + p_{0x} &= -\mu u'_0 \\
u_0 + p_{0y} &= -v'_0 \\
p_{0z} - b_0 &= 0 \\
u_{0x} + v_{0y} + w_{0z} &= 0 \\
0 &= -S'_0 \\
0 &= -T'_0 \\
b_0 &= T_0 - S_0.
\end{aligned} \quad (3.24\text{a–g})$$

The boundary conditions from (3.20) and (3.21) are

$$p_0(-1) = \eta_0 + \varphi$$
$$w_0(-1) = 0. \quad (3.25\text{a,b})$$

To solve the system in (3.24), we begin with the conservation laws for heat and salt in (3.24e) and (3.24f). These tell us that at leading order the temperature and salinity are independent of z. Thus we have

$$S_0 = \tilde{S}(x, y, t)$$
$$T_0 = \tilde{T}(x, y, t). \quad (3.26\text{a,b})$$

The vertical homogeneity of the leading-order temperature and salinity distribution justifies the description of the active layer as "mixed." The physical basis of (3.26) is that the approximation describes motion on timescales long relative to the vertical mixing timescale τ. Because the relaxation to vertical average is rapid, the leading-order temperature and salinity are vertically uniform.

From (3.24g), the leading-order buoyancy is

$$b_0 = \tilde{T} - \tilde{S} \equiv B(x, y, t), \quad (3.27)$$

and then using (3.24c) and (3.25a), the pressure is

$$p_0 = \eta_0 + \varphi + B(z + 1). \quad (3.28)$$

Next, the leading-order velocities are

$$\begin{aligned}
(\tilde{u}_0, \tilde{v}_0) &= \hat{\mathbf{z}} \times \nabla\psi_0 = (-\partial_y\psi_0, \partial_x\psi_0) \\
(u'_0, v'_0) &= (1 + \mu^2)^{-1}(z + 1/2) \\
&\quad \times [\hat{\mathbf{z}} \times \nabla B - \mu\nabla B] \\
w_0 &= \frac{\mu}{2}(1 + \mu^2)^{-1}(z^2 + z)\nabla^2 B,
\end{aligned} \quad (3.29\text{a–c})$$

where $\hat{\mathbf{z}}$ is the vertical unit vector and

$$\psi_0 \equiv \eta_0 + \varphi + (B/2) \quad (3.30)$$

is the streamfunction of the vertically averaged flow.

There is a nongeostrophic flow down that part pressure gradient associated with ∇B: this is the term $-\mu\nabla B$ on the right-hand side of (3.29b). However, the vertically averaged velocity is in geostrophic balance at leading order [see (3.29a)]. The gradient of the streamfunction ψ_0 in (3.30) is equal to the vertical average of the horizontal pressure gradient. From the right-hand side of (3.30) there are three contributions to this pressure gradient: displacements of the interface, pressure gradients in the fluid below the mixed layer, and a contribution from the buoyancy perturbations within the mixed layer. This final term does not appear in the QG approximation.

Notice that as μ becomes large, the vertically sheared part of the leading-order velocity in (3.29b) becomes small ($\sim\mu^{-1}$). This behavior is physically intuitive—if the vertical momentum mixing is large, then the vertical shears are weak. In this sense the slab mixed layer models are captured by the SML expansion as the limiting case $\mu \to \infty$.

One crucial point is that the vertical homogeneity of the leading-order temperature and salinity is ensured by $\epsilon \equiv \tau/\mathcal{T} \ll 1$ and the linear momentum balance is ensured by $\mathrm{Ro} \equiv 1/f\mathcal{T} \ll 1$. But these two inequalities do not imply that the velocity profile is slablike, even if $\tau \sim \tau_U$. For a slablike velocity one needs the more restrictive condition that $\mu \equiv 1/f\tau_U = \epsilon\mathcal{R}(\tau/\tau_U) \gg 1$; that is, the momentum mixing must be faster by order ϵ^{-1} than the stratification mixing.

e. *Terms of order ϵ^1*

Collecting the terms of order ϵ^1 in (3.14), one finds

$$
\begin{aligned}
-v_1 + p_{1x} &= -\mu u_1' \\
u_1 + p_{1y} &= -\mu v_1' \\
p_{1z} - b_1 &= 0 \\
u_{1x} + v_{1y} + w_{1z} &= 0 \\
\partial_{t_0} S_0 + u_0 S_{0x} + v_0 S_{0y} &= -S_1' \\
\partial_{t_0} T_0 + u_0 T_{0x} + v_0 T_{0y} &= -T_1' \\
b_1 &= T_1 - S_1. \qquad (3.31\text{a–g})
\end{aligned}
$$

The boundary conditions in (3.20) and (3.21) give

$$
\begin{aligned}
p_1(-1) &= \eta_1 \\
w_1(-1) &= 0. \qquad (3.32\text{a,b})
\end{aligned}
$$

To solve (3.31), we begin with the conservation equations for heat and salt in (3.31e) and (3.31f). Because $S_0 = \tilde{S}$ and $T_0 = \tilde{T}$ are independent of z, the vertical average of these conservation equations is

$$
\begin{aligned}
\partial_{t_0}\tilde{S} + J(\psi_0, \tilde{S}) &= 0 \\
\partial_{t_0}\tilde{T} + J(\psi_0, \tilde{T}) &= 0, \qquad (3.33\text{a,b})
\end{aligned}
$$

where $J(A, B) \equiv A_x B_y - A_y B_x = \hat{\mathbf{z}}\cdot\nabla A \times \nabla B$ is the Jacobian. Thus, at this order, the evolution of $\tilde{T}$ and $\tilde{S}$ is just advection by the vertically averaged velocity $(\tilde{u}_0, \tilde{v}_0) = (-\psi_{0y}, \psi_{0x})$.

Subtracting (3.33a,b) from (3.31e, f) gives the $\mathrm{O}(\epsilon^1)$ salinity and temperature

$$
\begin{aligned}
S_1 = S_1' &= (1 + \mu^2)^{-1}(z + 1/2)[\mathcal{J} + \mu\nabla B\cdot\nabla\tilde{S}] \\
T_1 = T_1' &= (1 + \mu^2)^{-1}(z + 1/2)[\mathcal{J} + \mu\nabla B\cdot\nabla\tilde{T}], \qquad (3.34\text{a,b})
\end{aligned}
$$

where

$$
\mathcal{J} \equiv J(\tilde{S}, \tilde{T}) = J(\tilde{S}, B) = J(\tilde{T}, B), \quad (3.35)
$$

is the Jacobian of $\tilde{T}$ with $\tilde{S}$. The fields T_1 and S_1 in (3.34) are created by the shear flow in (3.29b) tilting over the vertically homogeneous fields $\tilde{T}$ and $\tilde{S}$ in (3.26). The tilting is balanced by vertical mixing (Taylor 1953). The depth-dependent fields T_1 and S_1 in (3.34) are the result of this equilibration between the shear-driven tilting of $\tilde{T}$ and $\tilde{S}$ and the vertical mixing.

The expressions for T_1 and S_1 in (3.34a,b) give the $\mathrm{O}(\epsilon^1)$ buoyancy as

$$
b_1 = T_1 - S_1 = \mu(1 + \mu^2)^{-1}\left(z + \frac{1}{2}\right)\nabla B\cdot\nabla B, \quad (3.36)
$$

and then the hydrostatic balance (3.31c) is integrated to give

$$
p_1 = \frac{1}{2}\mu(1 + \mu^2)^{-1}(z^2 + z)\nabla B\cdot\nabla B, \quad (3.37)
$$

where the boundary condition in (3.32a) is satisfied by taking $\eta_1 = 0$. This amounts to requiring that the leading field η_0 contain all of the interface displacement.

Notice that (3.36) shows that at this order the mixed layer is stably stratified; that is, $b_{1z} > 0$.

Finally, using the momentum equations (3.31a,b), one can calculate the $\mathrm{O}(\epsilon^1)$ velocities:

$$
\begin{aligned}
\tilde{\mathbf{u}}_1 &= \hat{\mathbf{z}} \times \nabla\psi_1 \\
\mathbf{u}_1' &= \frac{1}{12}\mu(1 + \mu^2)^{-2}[\mathbf{z} \times \nabla(\nabla B\cdot\nabla B) \\
&\quad - \mu\nabla(\nabla B\cdot\nabla B)](6z^2 + 6z + 1) \\
w_1 &= \frac{1}{12}\mu^2(1 + \mu^2)^{-2}\nabla^2(\nabla B\cdot\nabla B)(2z^3 + 3z^2 + z), \\
&\qquad (3.38\text{a–c})
\end{aligned}
$$

where the $\mathrm{O}(\epsilon^1)$ streamfunction is

$$
\psi_1 \equiv -\frac{1}{12}\mu(1 + \mu^2)^{-1}\nabla B\cdot\nabla B. \quad (3.39)
$$

f. *Terms of order ϵ^2*

Collecting the terms of order ϵ^2 in (3.14), one has

$$
\begin{aligned}
\mathcal{R}[\partial_{t_0}u_0 + u_0 u_{0x} + v_0 u_{0y} + w_0 u_{0z}] - v_2 + p_{2x} &= -\mu u_2' \\
\mathcal{R}[\partial_{t_0}v_0 + u_0 v_{0x} + v_0 v_{0y} + w_0 v_{0z}] + u_2 + p_{2y} &= -\mu v_2' \\
p_{2z} - b_2 &= 0 \\
u_{2x} + v_{2y} + w_{2z} &= 0 \\
[\partial_{t_0}S_1 + u_0 S_{1x} + v_0 S_{1y} + w_0 S_{1z}] \\
+ [\partial_{t_1}S_0 + u_1 S_{0x} + v_1 S_{0y}] &= -S_2' \\
[\partial_{t_0}T_1 + u_0 T_{1x} + v_0 T_{1y} + w_0 T_{1z}] \\
+ [\partial_{t_1}T_0 + u_1 T_{0x} + v_1 T_{0y}] &= -T_2' \\
b_2 &= T_2 - S_2. \qquad (3.40\text{a–g})
\end{aligned}
$$

The boundary conditions (3.20) and (3.21) complete the $O(\epsilon^2)$ problem:

$$p_2(-1) - \eta_2 - \mathcal{R}\eta_0 b_0(-1) = 0$$

$$\mathcal{R}[\partial_{t_0}\eta_0 + u_0(-1)\eta_{0x} + v_0(-1)\eta_{0y}] + w_2(-1) - \mathcal{R}\eta_0 w_{0z}(-1) = 0. \quad (3.41\text{a,b})$$

Again we begin the solution of (3.40) by vertically averaging the conservation equations for heat and salt. We require that $\tilde{T}_1 = \tilde{S}_1 = 0$ so that the leading-order fields T_0 and S_0 carry all of the vertically averaged heat and salt. Notice also that $\tilde{T}'_2 = \tilde{S}'_2 = 0$ because the term proportional to η_0 in the final square bracket of (3.23b) is zero. With these simplifications the vertical average of (3.40e) is

$$\partial_{t_1}\tilde{S} + \nabla\cdot\widetilde{\mathbf{u}'_0 S'_1} + J(\psi_1, \tilde{S}) = 0. \quad (3.42)$$

Combining (3.33a) and (3.42) produces a reconstituted evolution equation for the vertically averaged salinity. Thus we write

$$\partial_t\tilde{S} = (\partial_{t_0} + \epsilon\partial_{t_1})\tilde{S} \quad (3.43)$$

and define

$$\psi \equiv \psi_0 + \epsilon\psi_1,$$
$$= \eta_0 + \varphi + \frac{1}{2}\left[B - \frac{1}{6}\epsilon\mu(1+\mu^2)^{-1}\nabla B\cdot\nabla B\right] + O(\epsilon^2). \quad (3.44)$$

Then evaluating the correlation $\widetilde{\mathbf{u}'_0 S'_1}$ using (3.29b) and (3.34a), finally gives

$$\partial_t\tilde{S} + J(\psi, \tilde{S}) = \frac{\epsilon}{12}(1+\mu^2)^{-2}\nabla\cdot[(\mathcal{J} + \mu\nabla B\cdot\nabla\tilde{S}) \times(\mu\nabla B - \hat{\mathbf{z}}\times\nabla B)] + O(\epsilon^2). \quad (3.45)$$

An analogous calculation gives the heat equation

$$\partial_t\tilde{T} + J(\psi, \tilde{T}) = \frac{\epsilon}{12}(1+\mu^2)^{-2}\nabla\cdot[(\mathcal{J} + \mu\nabla B\cdot\nabla\tilde{T}) \times(\mu\nabla B - \hat{\mathbf{z}}\times\nabla B)] + O(\epsilon^2). \quad (3.46)$$

The left-hand sides of the conservation equations (3.45) and (3.46) are exactly what one anticipates on the basis of a slab model: the mixed layer temperature and salinity are advected by the vertically averaged flow. The right-hand sides of (3.45) and (3.46) are nonlinear skew diffusivities; these terms are discussed in more detail in section 4.

Notice that if (3.45) is subtracted from (3.46) one has an equation for the buoyancy $B = \tilde{T} - \tilde{S}$. In this subtraction the terms containing $\mathcal{J} \equiv J(\tilde{S}, \tilde{T})$ in (3.45) and (3.46) cancel. It is obvious from the outset that T and S can be combined into a single equation for B. But (3.45) and (3.46) are nonlinearly coupled at $O(\epsilon^1)$ by the "$\mathcal{J}$ term." And in the important special case when $\mu \to 0$, the $\mathcal{J}$ terms are the only surviving coupling between $\tilde{S}$ and $\tilde{T}$ at $O(\epsilon^1)$.

The evolution equation for the geostrophic mode is now found with a slight variation of the familiar quasigeostrophic procedure. The momentum equations (3.36a) and (3.36b) are cross differentiated to eliminate p_2. The result of this is

$$\mathcal{R}[D_0\zeta_0 + v_{0z}w_{0x} - u_{0z}w_{0y} - w_{0z}\zeta_0] - w_{2z} = -\mu[\partial_x v'_2 - \partial_y u'_2], \quad (3.47)$$

where

$$\zeta_0 \equiv \partial_x v_0 - \partial_y u_0$$
$$D_0 \equiv \partial_{t_0} + u_0\partial_x + v_0\partial_y + w_0\partial_z$$
$$\mathbf{u}'_2 \equiv \mathbf{u}_2 - \tilde{\mathbf{u}}_2 - \mathcal{R}\eta_0[\mathbf{u}_0(-1) - \tilde{\mathbf{u}}_0]. \quad (3.48\text{a–c})$$

The term proportional to η_0 on the right-hand side of (3.48c) is analogous to that in the final square bracket in (3.22b): it arises because of the displacement of the lower boundary away from $z = -1$.

Integrating (3.47) from $z = -1$ to $z = 0$ gives

$$\mathcal{R}[\partial_{t_0}\tilde{\zeta}_0 + J(\psi_0, \tilde{\zeta}_0) + \nabla\cdot(\widetilde{\mathbf{u}'_0\zeta'_0} + \hat{\mathbf{z}}\times\widetilde{\mathbf{u}'_0 w_{0z}})] + w_2(-1)$$
$$= -\frac{1}{2}\mathcal{R}\mu(1+\mu^2)^{-1}[\mu J(B, \eta_0) + \nabla\cdot(\eta_0\nabla B)]. \quad (3.49)$$

The terms in the square bracket on the left-hand side of (3.49) are evaluated using (3.29). The term $w_2(-1)$ on the left-hand side of (3.49) is expressed in terms of quantities from the earlier orders in the expansion using (3.41b):

$$w_2(-1) = -\mathcal{R}\left[\partial_{t_0}\eta_0 + J(\psi_0, \eta_0) - \frac{1}{2}(1+\mu^2)^{-1} \times[J(B, \eta_0) - \mu\nabla\cdot(\eta_0\nabla B)]\right]. \quad (3.50)$$

In contrast to the standard quasigeostrophic approximation, the vertical velocity at $z = -1$ includes a contribution from the vertically sheared current in (3.23b) and a contribution from $\eta_0 w_{0z}$ in (3.16c). This is the origin of the square bracketed term on the right-hand side of (3.50).

Assembling all of this, (3.49) becomes the evolution equation for the vertical vorticity:

$$\partial_{t_0}[\zeta_0 - \eta_0] + J(\psi_0, \zeta_0) - J(\varphi, \eta_0) + \frac{1}{12}(1+\mu^2)^{-2} \times[(1-\mu^2)J(B, \nabla^2 B) - 2\mu\nabla\cdot(\nabla^2 B\nabla B)] = 0, \quad (3.51)$$

where $\zeta_0 \equiv \nabla^2\psi_0$, and

$$\eta_0 = \psi_0 - \varphi - \frac{1}{2}B \tag{3.52}$$

is the interface displacement.

g. The third-order terms

In (3.45) and (3.46) I derived the evolution equations of $\tilde{S}$ and $\tilde{T}$ correct to order $O(\epsilon^1)$. To obtain the same accuracy for the vertical vorticity, one must correct (3.51) by considering the third-order terms. Fortunately, a direct assault can be avoided by using the nondimensional version of the vertically averaged vorticity equation in (A.11). In terms of dimensionless variables, and with $w_{en} = \mathcal{F}_u = \mathcal{F}_v = 0$, this conservation law is

$$\partial_t(h\bar{\zeta} - \mathcal{R}^{-1}\epsilon^{-2}h) + \nabla\cdot[h(\bar{\mathbf{u}}\bar{\zeta} + \overline{\mathbf{u}'\zeta'} + \hat{\mathbf{z}}\times\overline{\mathbf{u}'w_z})] + \mathcal{R}^{-1}\epsilon^{-2}\Xi = 0, \tag{3.53}$$

where $h = 1 + \epsilon^2\mathcal{R}\eta$. One can substitute the earlier results into (3.53) and retain terms of $O(\epsilon^0)$ and $O(\epsilon^1)$. The only surviving term from Ξ is $\Xi = J(h, \hat{p}) = -\mathcal{R}\epsilon^2 J(\varphi, \eta)$.

The first simplification of (3.53) is that the distinction between $^{-}$ and $^{\wedge}$ appears only at $O(\epsilon^2)$ [see (3.22)]; thus, in (3.53) one can use $^{-}$ = $^{\wedge}$ + $O(\epsilon^2)$.

The second simplification in (3.53) is that

$$\widetilde{\mathbf{u}'\zeta'} + \hat{\mathbf{z}}\times\widetilde{\mathbf{u}'w_z} = \widetilde{\mathbf{u}_0'\zeta_0'} + \hat{\mathbf{z}}\times\widetilde{\mathbf{u}_0'w_{0z}} + O(\epsilon^2), \tag{3.54}$$

because the $O(\epsilon^1)$ terms vanish exactly; that is,

$$\widetilde{\mathbf{u}_1'\zeta_0'} = \widetilde{\mathbf{u}_0'w_{1z}} = 0, \quad \text{etc.} \tag{3.55}$$

[From (3.38b), $\mathbf{u}_1'$ is proportional to the polynomial $z^2 + z + 1/6 = (z + 1/2)^2 - (1/12)$ and $\mathbf{u}_0'$ is proportional to $z + 1/2$.] Thus the correlation in (3.53) is already completely calculated in (3.51): it is the term $(1/12)(1 + \mu^2)^{-2}[(1 - \mu^2)J(B, \nabla^2B) - 2\mu\nabla\cdot(\nabla^2B\nabla B)]$.

A third simplification in (3.53) is that

$$\hat{\zeta} = \nabla^2\psi + O(\epsilon^2), \tag{3.56}$$

where ψ is given by (3.44). This result follows from the nondimensional version of (A.3a): the final terms involving h_x and h_y are $O(\epsilon^2)$ because of (3.10). The fourth and final simplification is that since $\eta_1 = 0$, $\eta = \eta_0 + O(\epsilon^2)$.

Putting all of this together gives the corrected evolution equation for the vertical vorticity as

$$\zeta_t - \eta_t + J(\psi, \zeta) - J(\varphi, \eta) + \frac{1}{12}(1 + \mu^2)^{-2}[(1 - \mu^2)J(B, \nabla^2B) - 2\mu\nabla\cdot(\nabla^2B\nabla B)] + O(\epsilon^2) = 0. \tag{3.57}$$

In (3.57) the interface displacement is

$$\eta = \psi - \varphi - \frac{1}{2}B + \frac{1}{12}\epsilon\mu(1 + \mu^2)^{-1}\nabla B\cdot\nabla B, \tag{3.58}$$

and the condition that $\langle\eta\rangle = 0$ determines the constant of integration $\langle\psi\rangle$ that is needed for the inversion of $\zeta = \nabla^2\psi$.

The vertically averaged salinity and temperature evolve according to (3.45) and (3.46) and $B = T - S$. Thus, with (3.57), there are three nonlinearly coupled evolution equations for the five unknowns ψ, ζ, η, $\tilde{S}$, and $\tilde{T}$. Equation (3.58) and $\zeta = \nabla^2\psi$ provide two additional connections so that there is a closed evolutionary set. If $\tilde{S} = \tilde{T} = 0$, this set collapses to the usual quasigeostrophic equation for the vertical vorticity. We refer to the generalized set of low-frequency dynamics as the "subinertial mixed layer approximation" or the SML approximation.

4. Discussion

a. The SML approximation in dimensional variables

We begin our discussion of the SML approximation by summarizing the results from the previous section and translating them into dimensional variables. In this section we use $\bar{T}(x, y, t)$ and $\bar{S}(x, y, t)$ to denote the vertically averaged, dimensional temperature and salinity in the mixed layer. It is convenient to now define

$$\Theta \equiv g\alpha_T(\bar{T} - \check{T}), \quad \Sigma \equiv g\alpha_S(\bar{S} - \check{S}). \tag{4.1}$$

To leading order, the buoyancy is

$$B = \Theta - \Sigma, \tag{4.2}$$

so that the density is $\rho = \check{\rho}(1 - g^{-1}B + \cdots)$, where ellipses indicates the higher-order terms in the SML expansion [e.g., see (4.7)]. The reduced gravity g' is included as a large background constant in the buoyancy $B(x, y, t)$: $g' = \langle B\rangle$ where angle brackets denotes an area average.

The leading order horizontal velocities are

$$(u, v) = (-\psi_y, \psi_x) + f^{-1}(1 + \mu^2)^{-1}(z + \mathcal{H}/2) \times (-B_y - \mu B_x, B_x - \mu B_y), \tag{4.3}$$

where $\mu \equiv 1/f\tau_U$ and $\mathcal{H}$ is the average depth of the layer. Notice that the velocity field in (4.3) includes an ageostrophic component parallel to ∇B. This vertically sheared down–pressure gradient flow has an interesting dependence on μ: it has maximum amplitude when $\mu = 1$ and vanishes if either $\mu \to 0$ or $\mu \to \infty$. If μ is small then momentum mixing is weak and the ageostrophic velocity is small because of rotational constraints. But if μ is large, so that momentum mixing is strong, then the heavy friction retards the down-pressure gradient flow and again the ageostrophic velocity is small. The maximum ageostrophic response is achieved if $\mu = 1$.

The conservation laws for salt and heat are

$$\Sigma_t + J(\psi, \Sigma) = \frac{1}{12}(1+\mu^2)^{-2}(\tau\mathscr{H}^2/f^2)\,\nabla\cdot[(\mathcal{J} + \mu\nabla B\cdot\nabla\Sigma)(\mu\nabla B - \hat{\mathbf{z}}\times\nabla B)]$$

$$\Theta_t + J(\psi, \Theta) = \frac{1}{12}(1+\mu^2)^{-2}(\tau\mathscr{H}^2/f^2)\,\nabla\cdot[(\mathcal{J} + \mu\nabla B\cdot\nabla\Theta)(\mu\nabla B - \hat{\mathbf{z}}\times\nabla B)], \quad (4.4\text{a,b})$$

where τ is the mixing rate for T and S and $\mathcal{J} \equiv J(\Sigma, \Theta)$ is a Jacobian. The left-hand sides of (4.4) are intuitive: the vertically averaged flow advects the vertically averaged temperature and salt. The right-hand sides of (4.5) are skew, and cubically nonlinear, diffusivities due to the vertically sheared part of the velocity field in (4.3) (e.g., Taylor 1953). These terms are discussed in more detail in section 4c.

The vorticity equation is

$$\zeta_t - \eta_t + J(\psi, \zeta) - J(\psi, \eta) + \frac{1}{12}(\mathscr{H}/f)^2(1+\mu^2)^{-2}[(1-\mu^2)J(B, \nabla^2 B) - 2\mu\nabla\cdot(\nabla^2 B\nabla B)] = 0, \quad (4.5)$$

where $\zeta \equiv \nabla^2\psi$ and the location of the base of the mixed layer is $z = -\mathscr{H} - \mathscr{H}\eta/f$ with

$$\eta = \mathcal{L}^{-2}\psi - \mathcal{L}^{-2}\varphi - \frac{1}{2}(f/g')B + \frac{1}{12}\mu(1+\mu^2)^{-1}(\tau\mathscr{H}/g')\nabla B\cdot\nabla B. \quad (4.6)$$

In (4.5), $\mathcal{L} = \sqrt{g'\mathscr{H}}/f$ is the Rossby radius of deformation, and φ is essentially the pressure at the base of the mixed layer defined in (2.12). The integral constraints that $\langle\eta\rangle = 0$ and $\langle B\rangle = g'$ determine $\langle\psi\rangle$ from (4.6).

b. The slope of the density surfaces and the Richardson number

One distressing aspect of the evolution equations in (4.3)–(4.5) is their structural sensitivity to the size of $\mu \equiv 1/f\tau_U$. For instance, when $\mu = 1$ some coefficients change sign while others achieve their maximum values.

In this subsection I argue that the slope of the isopycnal surfaces within the mixed layer, or equivalently the vertical and horizontal buoyancy gradients, can be used to estimate μ. Unfortunately, with published data it is not possible to estimate these slopes, but nonetheless they might be extracted from sea soar observations, especially if repeated tracks were used to eliminate inertial oscillations.

In dimensional variables the reconstructed density field is

$$\rho = \check{\rho}\left\{1 - g^{-1}\left[B + (\tau/f)\mu(1+\mu^2)^{-1} \times \nabla B\cdot\nabla B\left(z + \frac{\mathscr{H}}{2}\right) + O(\epsilon^2)\right]\right\}. \quad (4.7)$$

The vertical structure of the mixed layer buoyancy is contained in the final, depth-dependent term, which is proportional to $\nabla B\cdot\nabla B$ and $z + (\mathscr{H}/2)$. The vertical structure is created by the shear flow in (4.3a) tilting over the vertically homogeneous field B. This process of shear dispersion is balanced by the vertical mixing, so that the depth-dependent term in (4.7) is proportional to τ: if τ is large then the vertical mixing is slow and the tilting proceeds further before it is checked. But the coefficient of the depth-dependent term is also proportional to $\mu(1+\mu^2)^{-1}$, which achieves its maximum value when $\mu = 1$. This factor comes directly from the dependence of $\mathbf{u}'\cdot\nabla B$ on μ [see (3.29b)]. If $\mu \ll 1$, then the tilting is weak because the shear flow is almost geostrophic and parallel to density surfaces. But if $\mu \gg 1$, then the shear flow is strongly nongeostrophic and also very weak because of the efficient vertical mixing of momentum. Thus, in either limit the depth-dependent part of the buoyancy is small and it happens that the maximum is found when $\mu = 1$.

The slope of the isopycnal surfaces is

$$s \equiv \frac{|\nabla\rho|}{\rho_z} \sim \frac{f}{\tau|\nabla B|}\left(\frac{1}{\mu} + \mu\right) \sim \frac{f^2}{|\nabla B|}\frac{1}{f\tau}\left(\frac{1}{\mu} + \mu\right). \quad (4.8)$$

Now, if the temperature changes by 3°C in 10^6 m and $\alpha_T \approx 3\times10^{-4}\,(°\text{C})^{-1}$, then $\nabla B = g\alpha_T\nabla T \sim 10^{-8}\,\text{s}^{-1}$. In this case the order of magnitude of the slope in (4.8) is determined by $(1/f\tau)(\mu + \mu^{-1})$.

The above estimates of isopycnal slope assumed that $\nabla B/f^2 \sim 1$ and this, in turn, was based on a temperature gradient of only 3°C/1000 km. This is a small gradient even for the broad-scale climatological distribution of temperature. On shorter length scales much larger values of ∇T have been observed. For instance, Price (1981) reports $\nabla T \sim$ 1°C/50 km in the wake of a hurricane and Samelson and Paulson (1988) found ∇T was as large as 2°C/3 km in the North Pacific Subtropical Frontal Zone. In these cases $f^2/|\nabla B| \ll 1$, and the slopes in (4.8) are correspondingly smaller. It is remarkable that (4.8) predicts an *inverse proportionality* between horizontal buoyancy gradients and isopycnal slopes.

One observable consequence of the theory is a relation between the vertical buoyancy gradient and the horizontal buoyancy gradient:

$$N^2 \equiv -\frac{g\rho_z}{\check{\rho}} = \frac{\tau}{\tau_U}(1+\mu^2)^{-1}f^{-2}\nabla B\cdot\nabla B. \quad (4.9)$$

The scaling $N^2 \sim f^{-2}\nabla B \cdot \nabla B$ might be tested with observations even though the constant of proportionality in (4.9) is uncertain. The relation (4.9) with $(\tau/\tau_U)(1+\mu^2)^{-2} = 1$ also emerges from the geostrophic adjustment model of Tandon and Garrett (1994). Thus, although the constant of proportionality is model dependent, the scaling $N^2 \sim f^{-2}\nabla B \cdot \nabla B$ is robust. The f^{-2} dependence suggests that the equatorial regions might be a site of strong shear-driven restratification and there is some observational evidence in support of this expectation (for example, Roemmich et al. 1994).

One can also calculate the Richardson number,

$$\mathrm{Ri} \equiv \frac{N^2}{u_z^2 + v_z^2}, \qquad (4.10)$$

of the subinertial motion. Using (3.29), one finds that in dimensional terms $u_z^2 + v_z^2 = (1+\mu^2)^{-1} f^{-2}\nabla B \cdot \nabla B$. The buoyancy frequency is in (4.7) so that the final result for the Richardson number is

$$\mathrm{Ri} = \frac{\tau}{\tau_U}. \qquad (4.11)$$

This simple expression for Ri comes from the cancellation of $\nabla B \cdot \nabla B$ between the numerator and denominator of (4.10). If one makes the natural assumption that momentum and density relax to their vertical averages at the same rate, then the Richardson number within the mixed layer is of order unity.

c. The nonlinear mixing of T and S: Thermohaline tilting

The conservation equations for heat and salt in (4.4) have a complicated nonlinear term on the rhs. Notice that one can linearly combine (4.4a,b) into a single equation for B and this results in the cancellation of $\mathcal{J} \equiv J(\Sigma, \Theta)$ from the system. But the symmetry between T and S might be broken by additional terms not explicitly included in (4.4). For instance, following Haney (1971), the atmospheric feedbacks on thermal anomalies might be modeled by a relaxation term such as $-\alpha\Theta$ in (4.4b). There is no reason to include an analogous term in (4.4a) and so the symmetry between T and S is lost. In this case one must deal with two independent tracers in the mixed layer (e.g., Stommel 1993).

If the momentum mixing is slower than the inertial period so that $\mu \equiv 1/f\tau_U \ll 1$, then the only surviving term on the rhs of (4.4a,b) is the proportional to $\nabla \cdot [\mathcal{J}\hat{\mathbf{z}} \times \nabla B]$ and this is the term that cancels in the subtraction used to obtain the buoyancy equation. The cancellation is to be expected: if $\mu \ll 1$ the vertically sheared velocity in (4.3) is in "thermal wind" or geostrophic balance. Consequently the shear lies in surfaces of constant B and does not tilt these same surfaces. But provided that the T and S fields are independent, so that $\mathcal{J} \neq 0$, the geostrophic shear does differentially advect the heat and salt surfaces. (This advection tilts the T and S surfaces so as not to create any buoyancy effect.) The fluxes of heat and salt associated with this "thermohaline tilting" mechanism are captured by the nonlinear term $\nabla \cdot [\mathcal{J}\hat{\mathbf{z}} \times \nabla B]$ on the rhs of (4.4a,b).

If the momentum mixing is rapid so that $\mu = 1/f\tau_U \gg 1$, then the rhs of (4.4a) is of order $\mu^{-2} \ll 1$. In this limit there is little transport of heat and salt by the terms on the rhs of (4.4a,b) because the strong vertical friction ensures that the sheared part of the velocity field is weak. The strongest surviving terms namely, $\mu^{-2}\nabla \cdot [\nabla B \cdot \nabla B \nabla \Sigma]$ and $\mu^{-2}\nabla \cdot [\nabla B \cdot \nabla B \nabla \Sigma]$, are nonlinear downgradient diffusion with a diffusivity proportional to the square of the buoyancy gradient. This particular nonlinearity is a well-known signature of buoyancy-driven shear dispersion [e.g., see the review by Young and Jones (1991) or Cessi and Young (1992)].

There are simple integral balances that help one understand the effect of the nonlinear terms on the rhs of (4.4a,b). First, both the advection term on the lhs and the term on the rhs are conservative so that $\langle \Sigma \rangle_t = \langle \Theta \rangle_t = 0$. Second, if one multiplies (4.3a) by Σ and (4.3b) by Θ and then takes the horizontal average the result is

$$\frac{1}{2}\langle \Sigma^2 \rangle_t = -\frac{1}{12}(1+\mu^2)^{-2}(\tau\mathscr{H}^2/f^2) \times \langle (\mathcal{J} + \mu\nabla B \cdot \nabla\Sigma)^2 \rangle \leq 0$$

$$\frac{1}{2}\langle \Theta^2 \rangle_t = -\frac{1}{12}(1+\mu^2)^{-2}(\tau\mathscr{H}^2/f^2) \times \langle (\mathcal{J} + \mu\nabla B \cdot \nabla\Theta)^2 \rangle \leq 0. \qquad (4.12\text{a,b})$$

[In taking all of these horizontal averages we assume that the boundary fluxes vanish: in a numerical simulation this might be ensured by using a doubly periodic representation of the fields.] The "power integrals" in (4.12) show that the temperature and salinity are both mixing toward their average levels $\langle \Theta \rangle$ and $\langle \Sigma \rangle$. In this sense the rhs of (4.4a,b) is nonlinear "downgradient" diffusion for all values of μ.

d. The vorticity equation

The buoyancy B effects the vertical vorticity through several different terms in (4.5) and (4.6).

First consider the limit $\mu = \infty$, so that the vertical momentum mixing is rapid. In this case one recovers the subinertial dynamics of slab ML models in which the only surviving buoyancy term in (4.5) and (4.6) is $(f/2g')B$ in the expression for the interface displacement η in (4.6). The case $\mu \gg 1$ is more subtle than the limit $\mu = \infty$ because the largest neglected terms are also among the most highly differentiated terms in the SML approximation. For instance, $\mu^{-1}\nabla B \cdot \nabla B$ in

(4.6) contains two spatial derivatives and $\mu^{-2}J(B, \nabla^2 B)$ in (4.5) contains four spatial derivatives. Thus the terms proportional to μ^{-1} and μ^{-2} will be important for disturbances with small spatial scales. It is difficult to make a general statement about the domain of validity in wavenumber space of the slab approximation. But in a specific problem, scale analysis based on comparing the retained term $(f/2g')B_t$ to the neglected terms, such as $(1/12)(\mathscr{H}/f)^2(1+\mu^2)^{-2}(1-\mu^2)J(B, \nabla^2 B)$, should roughly locate the high wavenumber validity boundary of the slab model.

The complementary limit is $\mu \to 0$ so that the sheared part of the velocity field in (4.3) is in geostrophic balance. In this case there is also substantial simplification of (4.5) and (4.6). But notice that the term $J(B, \nabla^2 B)$ in the vorticity equation (4.5) does survive if $\mu = 0$.

e. Some remarks on forcing

For simplicity, the derivation of the SML approximation in section 3 assumed that there was no forcing such as $(\mathscr{F}_u, \mathscr{F}_v)$ in the momentum equations and $\mathscr{F}_T$ and $\mathscr{F}_S$ in the temperature and salinity equations. This is a major simplification because it implies that all of the shear in the ML is due to the depth-dependent horizontal pressure gradient; for example, see (3.28) and (3.29). All of the vertical structure in the tracer fields is then attributable to differential advection by this same shear flow; for example, see (3.34).

But if the fluxes $\mathscr{F}_{\mathscr{X}}$ in (2.1) have depth dependence, this will be reflected in both the shear profile of the velocity field and in the vertical structure of the tracer fields. For example, see Niiler and Paduan (1994) where a parabolic stress profile, and a correspondingly sheared velocity profile, is shown to be consistent with drifter observations.

A complete discussion of this important point is beyond the scope of the present work. But there is some simple scale analysis that is reassuring because it shows that the ML velocities due to horizontal buoyancy gradients are at least comparable to the Ekman velocities.

If the surface wind stress is denoted by ρu_*^2, then the order of magnitude of the Ekman velocity is

$$\mathscr{U}_{\mathrm{Ek}} \equiv \frac{u_*^2}{f\mathscr{H}}. \tag{4.13}$$

On the other hand, from (4.3), the order of magnitude of the velocity due to horizontal buoyancy gradients is

$$\mathscr{U}_{\mathrm{BG}} \equiv \frac{\Gamma}{f\mathscr{H}}, \tag{4.14}$$

where $\Gamma \sim |\nabla B|$ is the order of magnitude of the horizontal buoyancy gradient. [In (4.14) we suppose that $\mu \sim 1$.] The ratio of the two velocities in (4.13) and (4.14) is a nondimensional number

$$\Upsilon \equiv \frac{\mathscr{U}_{\mathrm{BG}}}{\mathscr{U}_{\mathrm{Ek}}} = \frac{\Gamma\mathscr{H}^2}{u_*^2}, \tag{4.15}$$

measuring the relative importance of the Ekman flow (neglected throughout this paper) to the buoyancy-driven flow. Suppose that $u_* \sim 10^{-4}\ \mathrm{m^2\,s^{-2}}$ [corresponding to 1 dyn cm^{-2}], $\mathscr{H} \sim 100$ m, and $f \sim 10^{-4}\ \mathrm{s^{-1}}$. For the horizontal buoyancy we use the climatological value in (3.6): $\Gamma \sim 10^{-8}\ \mathrm{s^{-1}}$. With these numbers both $\mathscr{U}_{\mathrm{BG}}$ and $\mathscr{U}_{\mathrm{Ek}}$ are 1 cm s^{-1} and $\Upsilon = 1$.

The dependence of Υ on the square of the ML depth is notable: in deep subpolar mixed layers the sheared buoyancy gradient flow is much larger than the Ekman flow, whereas the reverse is likely the case in a shallow subtropical mixed layer at the end of summer. It is interesting that Υ is independent of f: as one approaches the equator both $\mathscr{U}_{\mathrm{Ek}}$ and $\mathscr{U}_{\mathrm{BG}}$ diverge at the same rate: f^{-1}.

These final cautionary remarks on Ekman transport emphasize the extreme idealization required to formulate the SML approximation. The approximations in this paper have isolated only one mechanism that might drive sheared velocity fields in the ML: these are the terms involving gradients of B in (4.3). It is interesting that the restratification created by this "∇B shear" is always statically stable as, for instance, in (4.7). In contrast, a sheared Ekman flow is not necessarily strongly correlated with the local ∇B. Thus it is possible that sheared Ekman transports might create statically unstable stratification, as in the process of mechanically forced convection.

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APPENDIX

The Vertically Averaged Vertical Vorticity Equation

This appendix contains the details of the vertical average of the vertical vorticity equation. The most important result of the manipulations in this appendix is the vertically averaged vertical vorticity (VAV^2) equation in (A.11).

Eliminating the pressure between (2.1a) and (2.1b) gives the vertical vorticity equation

$$\frac{D\zeta}{Dt} = fw_z + [(\zeta w)_z + (u_z w)_y - (v_z w)_x] + h^{-1}\mathscr{F}_\zeta - \tau_U^{-1}(v'_x - u'_y), \tag{A.1}$$

where

$$\zeta \equiv v_x - u_y, \quad \mathscr{F}_\zeta \equiv h[\partial_x(h^{-1}\mathscr{F}_v) - \partial_y(h^{-1}\mathscr{F}_u)]. \tag{A.2}$$

Construction of the vertical average of (A.1) requires care because the horizontal derivatives commute with neither ^ nor ¯. I use ∂_x and ∂_y, rather than subscripts, to denote derivatives in ambiguous cases. Using results such as $\partial_x\hat{u} = \partial_x u - h_x\partial_z u$, one can show

$$\bar{\zeta} \equiv \overline{\partial_x v} - \overline{\partial_y u} = \partial_x\bar{v} - \partial_y\bar{u} + h^{-1}[\hat{u}'h_y - \hat{v}'h_x]$$

$$\hat{\zeta} \equiv \partial_x v - \partial_y u = \partial_x\hat{v} - \partial_y\hat{u} + h_x\partial_z v - h_y\partial_z u$$

$$\zeta' \equiv \zeta - \bar{\zeta} = \partial_x v' - \partial_y u' - h^{-1}[\hat{u}'h_y - \hat{v}'h_x], \qquad \text{(A.3a–c)}$$

where $\hat{\mathbf{u}}'(x, y, t) = \mathbf{u}'(x, y, -h, t)$.

An additional useful identity is

$$\widehat{\frac{D\theta}{Dt}} = \hat{D}\hat{\theta} + w_{\text{en}}\partial_z\theta$$

$$\hat{D} \equiv \partial_t + \hat{u}\partial_x + \hat{v}\partial_y. \qquad \text{(A.4a,b)}$$

Substituting $\theta = z$ into (A.4a) reproduces the kinematic boundary condition in (2.6).

Using (2.7) the vertical average of (A.1) is

$$\partial_t(h\bar{\zeta}) + \nabla\cdot(h\overline{\mathbf{u}\zeta}) - w_{\text{en}}\hat{\zeta}$$
$$= -f\hat{w} + h\overline{\text{TILT}} + \bar{\mathcal{F}}_\zeta - \tau_U^{-1}(\hat{u}'h_y - \hat{v}'h_x), \qquad \text{(A.5)}$$

where

$$\overline{\text{TILT}} \equiv h^{-1}\int_{-h}^{0}(wu_z)_y - (wv_z)_x dz - h^{-1}\hat{w}\hat{\zeta} \qquad \text{(A.6)}$$

is the vertical average of the vortex tilting terms (the square bracket) in (A.1). The final term in (A.5), proportional to τ_U^{-1}, comes from using $\overline{\zeta'} = 0$ in (A.3c).

The vertical average of the vortex tilting terms, $\overline{\text{TILT}}$, can be manipulated into a more convenient form:

$$h\overline{\text{TILT}} = \partial_x(h\overline{v'w_z}) - \partial_y(h\overline{u'w_z}) + h\hat{v}'\partial_x(h^{-1}\hat{w})$$
$$- h\hat{u}'\partial_y(h^{-1}\hat{w}) - \bar{\zeta}\hat{w}. \qquad \text{(A.7)}$$

Substituting (A.7) into (A.5) and rearranging gives

$$\partial_t(h\bar{\zeta}) + \nabla\cdot h[\bar{\mathbf{u}}\bar{\zeta} + \overline{\mathbf{u}'\zeta'} + \hat{\mathbf{z}}\times\overline{\mathbf{u}'w_z}] + (f + \bar{\zeta})\hat{w}$$
$$+ h\hat{u}'\partial_y(h^{-1}\hat{w}) - h\hat{v}'\partial_x(h^{-1}\hat{w})$$
$$= w_{\text{en}}\hat{\zeta} + \bar{\mathcal{F}}_\zeta - \tau_U^{-1}(\hat{u}'h_y - \hat{v}'h_x). \qquad \text{(A.8)}$$

The advantage of (A.8) is that it easy to see how it reduces to the corresponding shallow water result. If there are no buoyancy variations within the layer then $\mathbf{u}' = 0$, $\hat{\zeta} = \bar{\zeta} = \partial_x v - \partial_y u$, and $\hat{w} = h\nabla\cdot\mathbf{u}$. Using these simplifications (A.8) collapses to

$$(h\zeta)_t + \nabla\cdot(h\mathbf{u}\zeta) + (\zeta + f)h\nabla\cdot\mathbf{u} = w_{\text{en}}\zeta + \mathcal{F}_\zeta, \qquad \text{(A.9)}$$

which is the flux divergence form of the shallow-water vertical vorticity equation.

Equation (A.8) is not the most convenient form for the expansion of section 3. Additional simplification can be made by eliminating the final term $\tau_U^{-1}(\hat{u}'h_y - \hat{v}'h_x)$. To do this, one first evaluates the horizontal momentum equations (2.1a,b) at $z = -h$ and then forms the linear combination h_x(2.1b) $- h_y$(2.1a). The result is

$$\tau_U^{-1}(\hat{u}'h_y - \hat{v}'h_x) = h_x\left[\widehat{\frac{Dv}{Dt}} + \partial_y\hat{p}\right] - h_y\left[\widehat{\frac{Du}{Dt}} + \partial_x\hat{p}\right]$$
$$- f[h_t + \hat{w} - w_{\text{en}}] - h^{-1}[h_x\hat{\mathcal{F}}_v - h_y\hat{\mathcal{F}}_u]. \qquad \text{(A.10)}$$

Substituting (A.10) into (A.8) and rearranging gives the vertically averaged vertical vorticity equation

$$\partial_t(h\bar{\zeta} - fh) + \nabla\cdot[h(\bar{\mathbf{u}}\bar{\zeta} + \overline{\mathbf{u}'\zeta'} + \hat{\mathbf{z}}\times\overline{\mathbf{u}'w_z})] + \Xi$$
$$= w_{\text{en}}[\hat{\zeta} - f] + h[\partial_x(h^{-1}\hat{\mathcal{F}}_v) - \partial_y(h^{-1}\hat{\mathcal{F}}_u)], \qquad \text{(A.11)}$$

where

$$\Xi \equiv h_x\left[\widehat{\frac{Dv}{Dt}} + \partial_y\hat{p}\right] - h_y\left[\widehat{\frac{Du}{Dt}} + \partial_x\hat{p}\right]$$
$$+ h\hat{u}'\partial_y(h^{-1}\hat{w}) - h\hat{v}'\partial_x(h^{-1}\hat{w}). \qquad \text{(A.12)}$$

Notice that (A.4a) might be used to rewrite the square brackets on the right-hand side of (A.12). For the purposes of the expansion in section 3, Eq. (A.11) is the most useful form of the vorticity conservation law.

Equation (A.11) is an exact consequence of (2.1a), (2.1b), (2.1d), and (2.6): the heat and salt conservation laws in (2.1e) and (2.1f) and the hydrostatic balance in (2.1c) have not been used in the manipulations leading from (2.19) to (A.11). The conservation law in (A.11) does not by itself provide a closed description of the dynamics. But (A.11) displays in a general and model-independent form the essential physics of subinertial vorticity dynamics. In section 3 a formal ordering scheme based on scale separation is used to develop a simplified version of (A.11).

REFERENCES

Allen, J. S., 1991: Balance equations based on momentum equations with global invariants of potential enstrophy and energy. *J. Phys. Oceanogr.*, **21**, 265–276.

Cessi, P., and W. R. Young, 1992: Multiple equilibria in two-dimensional thermohaline circulation. *J. Fluid Mech.*, **241**, 291–309.

Davis, R. E., R. de Szoeke, D. Halpern, and P. Niiler, 1981: Variability in the upper ocean during MILE. Part I: The heat and momentum balances. *J. Phys. Oceanogr.*, **28**, 1427–1451.

Dempsey, D. P., and R. Rotunno, 1988: Topographic generation of mesoscale vortices in mixed-layer models. *J. Atmos. Sci.*, **45**, 2961–2978.

de Szoeke, R. A., 1980: On the effects of horizontal variability of wind stress on the dynamics of the ocean mixed layer. *J. Phys. Oceanogr.*, **10**, 1439–1454.

——, and J. G. Richman, 1984: On wind-driven mixed layers with strong horizontal gradients—A theory with applications to coastal upwelling. *J. Phys. Oceanogr.*, **14**, 364–377.

Haney, R. L., 1971: Surface thermal boundary conditions for ocean circulation models. *J. Phys. Oceanogr.*, **1,** 241–248.

Lavoie, R. L., 1972: A mesoscale numerical model of lake-effect storms. *J. Atmos. Sci.*, **29,** 1025–1040.

McCreary, J. P., and P. K. Kundu, 1988: A numerical investigation of the Somali Current during the southwest monsoon. *J. Mar. Res.*, **46,** 1–23.

——, and Z. Yu, 1992: Equatorial dynamics in a 2½-layer model. *Progress in Oceanography,* Vol. 29, Pergamon, 61–132.

McWilliams, J. C., and P. R. Gent, 1980: Intermediate models of planetary circulations in the atmosphere and ocean. *J. Atmos. Sci.*, **37,** 1657–1678.

Niiler, P. P., and J. D. Paduan, 1994: Wind-driven motions in the northeast Pacific as measured by Lagrangian drifters. *J. Phys. Oceanogr.*, **24,** submitted.

Pedlosky, J. P., 1987: *Geophysical Fluid Dynamics.* Springer-Verlag, 710 + xiv pp.

Price, J., 1981: Upper ocean response to a hurricane. *J. Phys. Oceanogr.*, **11,** 153–175.

Ripa, P., 1993: Conservation laws for primitive equation models with inhomogeneous layers. *Geophys. Astrophys. Fluid Dyn.*, **70,** 85–111.

Roemmich, D., M. Y. Morris, W. R. Young, and J. R. Donguy, 1994: Fresh equatorial jets. *J. Phys. Oceanogr.*, **24,** 540–558.

Salmon, R., 1985: New equations for nearly geostrophic flow. *J. Fluid Mech.*, **153,** 461–477.

Samelson, R. M., and C. A. Paulson, 1988: Towed thermistor chain observations of fronts in the subtropical North Pacific. *J. Geophys. Res.*, **93,** 2237–2246.

Schopf, P. S., and M. A. Cane, 1983: On equatorial dynamics, mixed layer physics and sea surface temperature. *J. Phys. Oceanogr.*, **13,** 917–935.

Stommel, H. M., 1993: A conjectural regulating mechanism for determining the thermohaline structure of the oceanic mixed layer. *J. Phys. Oceanogr.*, **23,** 142–148.

Tandon, A., and C. J. R. Garrett, 1994: On mixed layer restratification due to a horizontal density gradient. *J. Phys. Oceanogr.*, **24,** in press.

Taylor, G. I., 1953: Dispersion of soluble matter in solvent flowing slowly down a tube. *Proc. Roy. Soc. London Ser. A,* **219,** 186–203.

Young, W. R., and S. Jones, 1991: Shear dispersion. *Phys. Fluids A,* **3,** 1087–1101.