# Selective Decay of Enstrophy and the Excitation of Barotropic Waves in a Channel

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ABSTRACT

If the momentum, energy and circulation of a fluid in a periodic, quasi-geostrophic, $\beta$-plane channel are specified, then there is a minimum enstrophy implied. This minimum enstrophy flow is obtained using the calculus of variations and is found to be also a solution of the quasi-geostrophic equations. It is either a parallel flow or a finite-amplitude Rossby wave, depending on the aspect ratio of the channel and the amount of energy and momentum within it. The most geophysically relevant case is a channel whose zonal length is substantially greater than its meridional breadth. In this instance the form of the minimum enstrophy solution is decided by the ratio of the energy to the squared momentum. When this parameter is below a critical value one has a parallel flow, while if this value is exceeded, the minimum enstrophy solution is a Rossby wave.

Heuristic arguments based on the enstrophy cascade in two-dimensional turbulence suggest a "selective decay hypothesis". This is that scale-selective dissipation will decrease the enstrophy more rapidly than the energy, momentum and circulation. If this is the case, then the system should approach the minimum enstrophy solution.

## 1. Introduction

Hou and Farrell (1986) have recently obtained a class of exact, nonlinear, stationary solutions for barotropic waves in a $\beta$-plane channel. Using numerical experiments they show that these waves can develop from a variety of initial conditions. However, not all initial conditions give rise to the wave; rather, if a persistent wave is to appear, a critical energy must be exceeded. If the initial energy is less than the critical value, then the fluid settles into a parallel shear flow. The present note discusses the emergence of these finite-amplitude waves from a general initial condition using a "selective decay principle" (Bretherton and Haidvogel, 1976; Matthaeus and Montgomery 1980; Leith, 1984). The essential idea is that the many invariants of the inviscid barotropic potential vorticity equation evolve on different time scales when scale-selective dissipation acts. For a $\beta$-plane channel the invariants are energy, zonal momentum and an infinite number of generalized enstrophy integrals. The circulation in the channel is the simplest of these enstrophy invariants. Scale-selective viscosity alters these invariants and it is plausible that the enstrophy invariants, which have a larger contribution from high wavenumbers, are changed more rapidly than the energy and momentum. This suggests that with scale-selective dissipation the flow initially evolves into a state which has the smallest enstrophy which is consistent with the initial energy, circulation and momentum. On longer time scales the momentum, circulation and energy are gradually altered so that, in the absence of external forcing, the ultimate state is one of rest.

The heuristic argument above suggests a variational principle: minimize one of the enstrophy integrals, subject to the constraints that the energy, circulation and momentum are constant. For tractability the quadratic enstrophy is minimized in the calculation below. The result is that in a channel with realistic aspect ratio, if the energy is sufficiently large, or the momentum small, then the minimum enstrophy solution is the nonlinear barotropic wave discussed by Hou and Farrell (1986). Alternatively, for small energy or large momentum, the minimum enstrophy solution is a parallel flow.

It should be clear at the outset that the selective decay principle, as stated above, is a heuristic tool which is based on plausible assumptions about the enstrophy and energy cascade in two-dimensional turbulence. As a predictor of the evolution of arbitrary initial conditions it is not rigorously deductive and is sometimes wrong (e.g., Bretherton and Haidvogel's experiment 2). Nonetheless, the variational calculation described here is still a means of establishing some important landmarks in parameter space. It is useful to know in advance that if one prescribes the energy, momentum and circulation of a fluid then the enstrophy must be larger than a certain minimum. The amount by which it exceeds this minimum (the "excess enstrophy") is an indication of how far the enstrophy cascade can possibly proceed before it is interrupted by the momentum, energy and circulation invariants.

## 2. The equations of motion and their invariants

The inviscid barotropic potential vorticity equation in a $\beta$-plane channel is

$$q_t + J(\psi, q) = 0$$
$$q = \beta y + \nabla^2\psi, \qquad (2.1)$$

with boundary conditions at the channel walls

$$\psi_x(x, \pm b) = 0$$
$$\bar{u}(\pm b) = \pm U \qquad (2.2)$$

where the overbar is an $x$ average and $U$ is a constant velocity. We suppose that the flow is periodic

$$\psi(x, y) = \psi(x + a, y) \qquad (2.3)$$

with zonal wavelength $a$. One important point to note is that there is a special choice of reference frame in (2.2b). Conservation of circulation ensures that the zonally averaged velocity at each wall is a constant. But there is no loss of generality in adopting a frame of reference in which the two wall velocities are equal in magnitude and oppositely directed.

From the previous equations it is straightforward to prove that energy

$$E \equiv \int \nabla\psi \cdot \nabla\psi dA/2, \qquad (2.4)$$

and zonal momentum

$$M \equiv \int y\nabla^2\psi dA = \int u dA, \qquad (2.5)$$

are constant. Additionally, there are an infinite number of enstrophy invariants

$$Q = \int F(q) dA \qquad (2.6)$$

where $F$ is an arbitrary function. One important special choice is $F(q) = q$, which shows that the circulation in the channel is constant and indeed this is equivalent to the boundary condition (2.2b). Also of particular importance to the present work is the quadratic enstrophy invariant

$$Z \equiv \int (\nabla^2\psi)^2 dA/2 \qquad (2.7)$$

which is obtained by subtracting a multiple of (2.5) from (2.6) with $F(q) = q^2/2$.

It is convenient to adopt nondimensional variables, temporarily denoted by an asterisk:

$$(x, y) = b(x_*, y_*)$$
$$\psi = \beta b^3 \psi_*$$
$$q = \beta b q_* = \beta b(y_* + \nabla_*^2\psi_*). \qquad (2.8)$$

The nondimensional invariants are defined as

$$a_* E_* \equiv \int \nabla_*\psi_* \cdot \nabla_*\psi_* dA_*/2$$
$$a_* M_* = \int y_* \nabla_*^2 \psi_* dA_* = -\int \psi_{*y_*} dA_*$$
$$a_* Z_* \equiv \int (\nabla_*^2\psi_*)^2 dA/2 \qquad (2.9)$$

where $a_* = a/b$ is the nondimensional channel length.

Before turning to the variational problem we list some direct inequalities which follow from (2.9) and the Cauchy-Schwarz inequality:

$$\int f^2 dA \int g^2 dA \geqslant \left\{\int fg dA\right\}^2. \qquad (2.10)$$

First from (2.9a) with $f = \psi_y$ and $g = 1$:

$$4a_*^2 E_* \geqslant \int \psi_y{}^2 dA \int 1 dA \geqslant a_*^2 M_*^2$$

or

$$e \equiv E_*/M_*^2 \geqslant \frac{1}{4}. \qquad (2.11)$$

Likewise from (2.9c), with $f = \nabla^2\psi$ and $g = y$,

$$z \equiv Z_*/M_*^2 \geqslant \frac{3}{4}. \qquad (2.12)$$

The result (2.11) is important because it tells us that even though the energy and momentum can be specified independently, $e$ must exceed ¼. Likewise, the variational bound on the enstrophy can do no better than (2.12); i.e., one anticipates that the minimum enstrophy is greater than $(3/4)M_*^2$, whatever $E_*$ and $M_*$.

## 3. The variational problem and its solution

The minimum enstrophy, subject to constant energy and momentum as constraints, is found by minimizing the functional:

$$F[\psi] \equiv \int \frac{1}{2}(\nabla^2\psi)^2 dA + \delta \int \frac{1}{2}\nabla\psi \cdot \nabla\psi dA + \epsilon \int y\nabla^2\psi dA \qquad (3.1)$$

where $\delta$ and $\epsilon$ are Lagrange multipliers which are determined later. In constructing the functional, it is assumed that $\psi$ satisfies the same boundary and periodicity conditions as the solution of the equations of motion, i.e., (2.2) and (2.3). This ensures that the total circulation in the channel is included as a constraint without adding a third Lagrange multiplier to (3.1). The Euler-Lagrange equation, which is a *necessary* condition for a minimum, is

$$\nabla^2\psi - \delta\psi + \epsilon y = 0. \qquad (3.2)$$

The preceding equation must be solved with the boundary conditions

$$\psi_x(x, \pm 1) = 0$$
$$-\bar{\psi}_y(\pm 1) = \pm U$$
$$\psi(x+a, y) = \psi(x, y). \tag{3.3}$$

One then calculates the energy and momentum and so finds $\epsilon$ and $\delta$. Finally, perhaps by examining the second variation, one can determine whether the stationary solution constructed above is actually a minimum.

Before discussing the detailed solution of (3.2), we note that it implies

$$q = \nabla^2\psi + y = \delta\{\psi + cy\}$$
$$c \equiv (1-\epsilon)/\delta \tag{3.4}$$

and solutions of the potential vorticity equation with this functional relation between potential vorticity and streamfunction are waves with phase speed $c$. [To see this, look for finite-amplitude solutions of (2.1) which have the form $\psi(x - ct, y)$.] Equation (3.4) is precisely the relation which underlies the shape-preserving, nonlinear Rossby waves discussed by Hou and Farrell (1986). Thus, their solutions are stationary (but perhaps not extremal) points of the functional (3.1). However, they did not discuss the solution of (3.2) from the perspective of variational calculus and the remainder of this section is devoted to this task.

It is remarkable that the solution of the variational problem is also a special solution of the quasi-geostrophic equations. There was no guarantee of this at the outset, and indeed there are many examples in the literature where the functions which extremize some quantity subject to constraints are not solutions of the dynamics which motivated the variational problem (e.g., Howard, 1972). Thus, not only is there a minimum enstrophy for given $E$, $M$ and $U$, but this minimum can actually be achieved by solutions of the quasi-geostrophic equations. Moreover, in an *ideal* fluid, these solutions must be stable because all the adjacent states have more enstrophy and so are inaccessible if enstrophy is conserved.

*a. Symmetric solutions of the Euler-Lagrange equation*

We first look for symmetric (i.e., independent of $x$) solutions of (3.2). In this instance, one is solving an ordinary differential equation and the result is

$$u = -\psi_y = \frac{\epsilon}{\gamma^2}\left\{1 - \frac{\cos\gamma y}{\cos\gamma}\right\} + U\frac{\sin\gamma y}{\sin\gamma} \quad \text{if } \delta = -\gamma^2 < 0,$$
$$u = -\psi_y = \frac{\epsilon}{\alpha^2}\left\{1 - \frac{\cosh\alpha y}{\cosh\alpha}\right\} + U\frac{\sinh\alpha y}{\sinh\alpha} \quad \text{if } \delta = \alpha^2 > 0. \tag{3.5}$$

With another integration one can obtain $\psi$, but this is not necessary. After some straightforward albegra one finds the energy, momentum and quadratic enstrophy defined in (2.9). The explicit formulas are presented in the Appendix.

*b. Nonsymmetric solutions of the Euler–Lagrange equation*

Nonsymmetric solutions of (3.2) have the form

$$\psi = \bar{\psi}(y) + \psi'(x - ct, y) \tag{3.6}$$

where

$$\bar{\psi}_{yy} - \delta\bar{\psi} + \epsilon y = 0$$
$$\psi'_{xx} + \psi'_{yy} - \delta\psi' = 0 \tag{3.7}$$

and $c$ is given by (3.4b). If $\psi'$ is to satisfy (3.3a, c) then $\delta$ must be quantized

$$-\delta = \gamma_{mn}{}^2 = (2\pi m/a)^2 + (n\pi/2)^2 \tag{3.8}$$

where $m$ and $n$ are positive integers. If $n$ is odd

$$\psi' = \phi\cos(n\pi y/2)\cos\{2\pi m(x - ct)/a\}, \tag{3.9}$$

while if $n$ is even, $\psi'$ is proportional to $\sin(n\pi y/2)$. In (3.9), $\phi$ is an amplitude which will be determined subsequently.

For the moment confine attention to the lowest wavenumber

$$\gamma_{11} \equiv [(\pi/2)^2 + (2\pi/a)^2]^{1/2} \tag{3.10}$$

and the corresponding candidate for a nonsymmetric, minimum enstrophy solution is the gravest mode

$$\psi' = \phi\cos(\pi y/2)\cos[2\pi(x - ct)/a]$$
$$\bar{u} = -\bar{\psi}_y$$
$$= (\epsilon/\gamma_{11}{}^2)\left\{1 - \frac{\cos\gamma_{11}y}{\cos\gamma_{11}}\right\} + U\frac{\sin\gamma_{11}y}{\sin\gamma_{11}}$$

where $\gamma_{11}$ is given by (3.10).

The final step is to determine $\epsilon$ and $\phi$ using the known values of energy and momentum. All of the momentum resides in the mean flow component. Thus,

$$M = \int_{-1}^{1} \bar{u}dy$$
$$= 2(\epsilon/\gamma_{11}{}^2)\{1 - \gamma_{11}{}^{-1}\tan\gamma_{11}\} \tag{3.11}$$

and this is a simple connection between $\epsilon$ and $M$.

The energy is

$$E = \bar{E}(\gamma_{11}) + E'$$
$$E' = \phi^2\gamma_{11}{}^2/4 \tag{3.12}$$

where $\bar{E}(\gamma_{11})$ is defined by (A1a). The expression $\gamma_{11}$ is a known constant and the preceding is essentially a simple relation between the known total energy $E$ and the amplitude $\phi$. As Hou and Farrell (1986) noted, there is a minimum value of $E$, $E(\bar{\gamma}_{11})$, which must be exceeded if the wave is to exist. Finally, one calculates the enstrophy of the nonsymmetric solution and relates it to the energy:

$$Z = \bar{Z}(\gamma_{11}) + \gamma_{11}^2[E - \bar{E}(\gamma_{11})] \tag{3.13}$$

where $\bar{Z}(\gamma_{11})$ is defined in (A1c).

### c. *Summary of the solution of Euler–Lagrange equation*

The Euler–Lagrange equations have both symmetric, (3.5), and nonsymmetric, (3.6)–(3.9), solutions. While these are stationary points of the functional (3.1) we have still to decide if they are extrema and whether the minimum enstrophy is achieved by a symmetric or nonsymmetric flow. This is the focus of the next section. However, it may help to anticipate (or even avoid) the detailed discussion which is given there. This is done schematically in Fig. 1, which assumes that both $U$ and $M$ are nonzero constants. (The important cases of zero $M$ or $U$ require separate discussion.)

Figure 1 shows the energy–enstrophy relation which results from eliminating the Lagrange multipliers. The solid, multibranched curve is energy–enstrophy relationship of the symmetric solutions in (3.5). The Lagrange multiplier, $\delta$, is a parameter which moves along this curve. Thus, the lowest point (B) corresponds to $\delta = 0$. The extreme left-hand section (AB) has $\delta = \alpha^2 > 0$. The remainder of the lowest solid branch (BCE) corresponds to $-\pi^2 \leqslant \delta = -\gamma^2 < 0$. The upper, cusped branches, which are clearly not minimum enstrophy solutions, have $\gamma > \pi$.

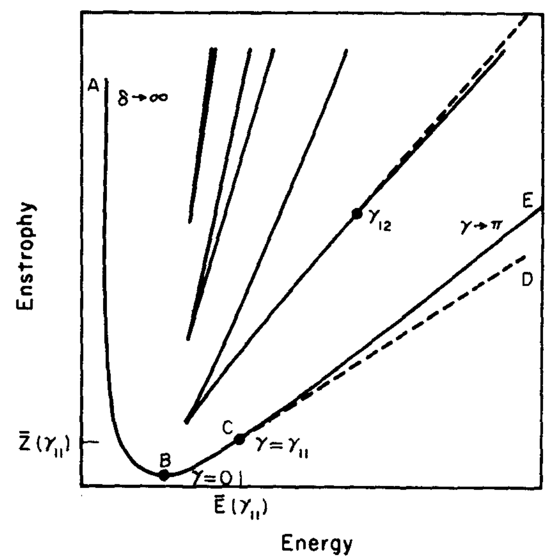


FIG. 1. A schematic of the energy–enstrophy relations. The solid, multibranched curve is the symmetric solution in (3.5), while the dashed straight lines are the first two nonsymmetric solutions in (3.6), (3.9) and (3.12). The Lagrange multiplier, $\gamma$ or $\alpha$, is a parameter which "moves along" the solid curve, e.g., the lowest point B is $\gamma = 0$.

The dashed lines, which are tangent to the solid curve, are the energy–enstrophy relations of the first two nonsymmetric modes, $\gamma_{11}$ and $\gamma_{12}$. Unlike the symmetric case, there is a simple analytic expression for this relation, (3.13). This is a consequence of the different character of the two solutions. In the symmetric case $\delta$ is a continuously varying parameter, whereas in the nonsymmetric case it is "quantized" as in (3.8). Instead, the amplitude of the wave, $\phi$, is a continuous variable whose elimination gives (3.13).

In Fig. 1 it is assumed that $\gamma_{11}$ is less than $\pi$ so that the nonsymmetric solution bifurcates from the symmetric on the lowest branch, BCE. This is the case if the length of the channel, $a$, is sufficiently large. In this eventuality the minimum enstrophy solution is ABCD. Thus, when the energy is less than $\bar{E}(\gamma_{11})$ it is a parallel flow, while if this treshold is exceeded it is a parallel flow with a wave superimposed.

Of course, if $a$ is reduced then $\gamma_{11}$ is eventually larger than $\pi$ and all of the nonsymmetric solutions bifurcate from one of the upper, cusped branches. In this case the minimum enstrophy solution is a parallel flow even when the energy is large.

## 4. Details of the variational solution

### a. *Special case (i): $M = 0$ and $U = 0$*

The first special case is when both the momentum and the circulation are zero. Consider first the symmetric solutions. There are two of these.

First, with $U = 0$ and $\epsilon \neq 0$ in (3.5a), one has the profile

$$u(y) = (\epsilon/\gamma_*^2)\left(1 - \frac{\cos\gamma_* y}{\cos\gamma_*}\right), \tag{4.1}$$

where $\gamma_*$ is determined by requiring that the momentum in (A1b) is zero. Thus $\gamma_*$ is the first nontrivial zero of $\gamma = \tan\gamma$ or

$$\gamma_* = 4.4934 \cdots. \tag{4.2}$$

The velocity profile in (4.1) is shown in Fig. 2.

The second symmetric solution with zero momentum is obtained by first setting $\epsilon = 0$ in (3.5a) and then taking a singular limit, $U \to 0$ and $\gamma \to \pi$ but holding the ratio $U/\sin\gamma$ fixed at $U_0$. The result is

$$u(y) = U_0 \sin\pi y, \tag{4.3}$$

which satisfies both the boundary conditions and the zero momentum constraint. Again the velocity profile is shown in Fig. 2.

Figure 3 shows the enstrophy–energy relations of both (4.1) and (4.3). Clearly (4.3) has then smaller enstrophy for given energy.

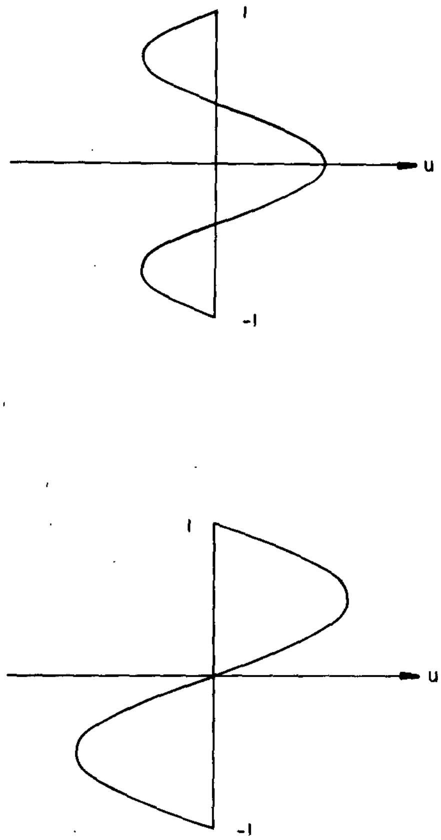


FIG. 2. The velocity profiles corresponding to the stationary enstrophy solutions in (4.1) and (4.3). For fixed energy, the lower profile always has less enstrophy than the upper. However, if the inequality (4.5) is satisfied then there is a nonsymmetric solution with less enstrophy than the lower profile.

The gravest nonsymmetric solution is

$$\psi = \phi \cos(\pi y/2) \cos[\pi(x - ct)/a] \tag{4.4}$$

and its wavenumber is

$$\gamma_{11} = [(\pi/2)^2 + (2\pi/a)^2]^{1/2}.$$

With fixed energy, the solution in (4.4) has less enstrophy than (4.3) provided $\gamma_{11}$ is less than $\pi$, or equivalently,

$$a > 2.31. \tag{4.5}$$

Figure 3 shows the enstrophy–energy relation of the gravest mode in a channel with $a = 12$. (Recall the nondimensional north–south width of the channel is 2 so this choice is a region whose zonal length is six times its meridional.) Thus, in this case, the minimum enstrophy solution is the Rossby wave in (4.4). Note, however, that if the channel is "square" ($a = 2$), the minimum enstrophy solution is the parallel flow in (4.3).

Also shown in Fig. 3 is the enstrophy–energy relation of a "modon sea" (Stern, 1975). This is an ensemble of close-packed modons of radius one, which is the largest size permitted by a channel with $a > 2$. Smaller modons have even steeper enstrophy–energy relations.

*b. Special case (ii): $M = 0$ and $U \neq 0$*

The previous case was doubly special because both $U$ and $M$ were zero. Now suppose that the momentum is still zero but the circulation is nonzero. This characterizes Hou and Farrell's (1986) numerical experiments.

First, consider symmetric solutions. From (A1b) and (A2b) if there is no momentum, then either

(a) $\epsilon = 0$ and $\gamma$ is a continuous variable
(b) $\gamma = \gamma_* = 4.4934 \cdots$.

(We ignore the larger solutions of $\gamma = \tan\gamma$ because these are clearly not minimum enstrophy candidates.) In case (a) the results are summarized in Fig. 4, which shows $Z/U^2$ as a function of $E/U^2$. This is obtained by the elimination of $\gamma$ or $\alpha$ from the expressions given in the Appendix. Thus, $\delta$ ($=\alpha^2$ or $-\gamma^2$) parameterizes the solid curve in Fig. 4. The lowest point of the curve is $\delta = 0$. The left-hand branch, which has a vertical asymptote ($Z \to \infty$ as $E \to 0$) is $\delta = \alpha^2 > 0$. The right-hand branch, which asymptotes to $Z \sim \pi^2 E + O(E^{1/2})$, is $\pi^2 > \gamma^2 = -\delta > 0$. Figure 1 schematically shows that there are many disjoint, cusped branches when $\gamma^2 > \pi^2$. Only one of these is indicated in Fig. 4. Figure 5 shows the velocity profiles which correspond to various indicated points on Fig. 4. As $E/U^2$ becomes large, the velocity profile becomes increasingly similar to (4.3), and thus case (i) is recovered.

The solid curve in Fig. 4 is the enstrophy–energy

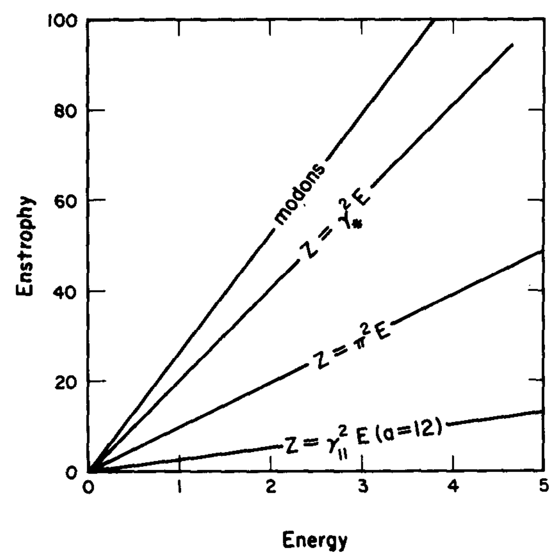


FIG. 3. Energy enstrophy relations when $U = M = 0$. In this case the curves in Fig. 1 degenerate into straight lines. When $a = 12$ the minimum enstrophy solution is the gravest mode $\gamma_{11}$.

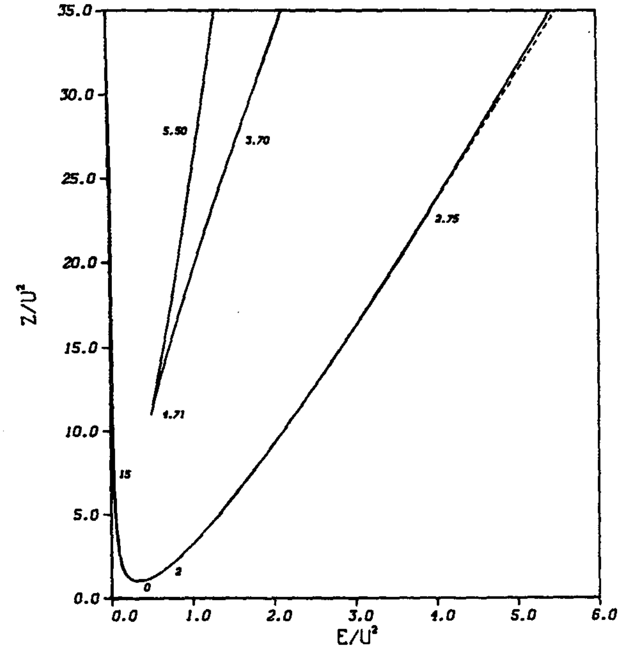


FIG. 4. The solid curves show the energy–enstrophy relation of symmetric solutions in (3.5) with $\epsilon = 0$ and $\gamma$ or $\alpha$ continuously varying. Selected values of $\gamma$ or $\alpha$ are shown on the plot. The dashed straight line is the energy-enstrophy relation [see (3.13)] of the gravest nonsymmetric mode in a channel with $a = 12$. It is tangent to the solid curve [see (4.6)] at $\gamma = \gamma_{11} \approx 2.75$.

relation of a symmetric flow. The enstrophy–energy relations of the nonsymmetric modes, which from (3.13) are just straight lines, bifurcate from the solid curve when $\gamma = \gamma_{mn}$. Shown in Fig. 4 as a dashed straight line is the relation for the gravest mode when $a = 12$ and $\gamma_{11} = 2.741$. This straight line is tangent to the solid curve because

$$(\partial \bar{Z}/\partial \bar{E})_{U,M} = -\delta = \gamma^2. \tag{4.6}$$

This important relation can be tediously verified by direct differentiation of the results in the Appendix. Alternatively, one can observe that because of the underlying variational problem the difference between the enstrophy of the symmetric solution and that of the nonsymmetric must be of second order in the difference between these two fields. Thus, the two curves must be tangent at the bifurcation point. In any case, provided the inequality (4.5) is satisfied, the nonsymmetric mode branches from the lowest curve and so if $E/U^2$ is greater than $E(\gamma_{11})/U^2$, the minimum enstrophy solution is a wave.

However, Hou and Farrell (1986) used a "square channel", i.e., $a = 2$. In their case $\gamma_{11} = 1.12\pi$ so that the gravest nonsymmetric bifurcates from one of the upper, cusped branches. The persistent wave which develops in their simulations is not a minimum enstrophy solution—there is a parallel flow with the same energy, momentum and circulation, but less enstrophy. Possible reasons for the failure of the selective decay principle are discussed in the conclusion.

So far our discussion has ignored the second symmetric solution, (b), with $\gamma = \gamma_*$. The enstrophy–energy relation of this mode is easily obtained by simplifying the expressions in the Appendix using $(\tan)\gamma_* = \gamma_*$. The result is

$$(Z/U^2) = \gamma_*^2(E/U^2) + 1. \tag{4.7}$$

Like the nonsymmetric solutions, this is a straight line which is tangent to the solid curve in Fig. 4 at $\gamma = \gamma_*$. The point of bifurcation or tangency lies on the upper, cusped branch because $\gamma_* > \pi$. Thus, there is always either another parallel flow, or a wave, with less enstrophy for given energy and circulation. The linear enstrophy–energy relation of this mode is not shown in Fig. 4.

*c. Special case (iii): $M \neq 0$ and $U = 0$*

Consider the complementary case where the circulation vanishes but the momentum is nonzero. Again we begin by examining the symmetric solution and noting there are two types:

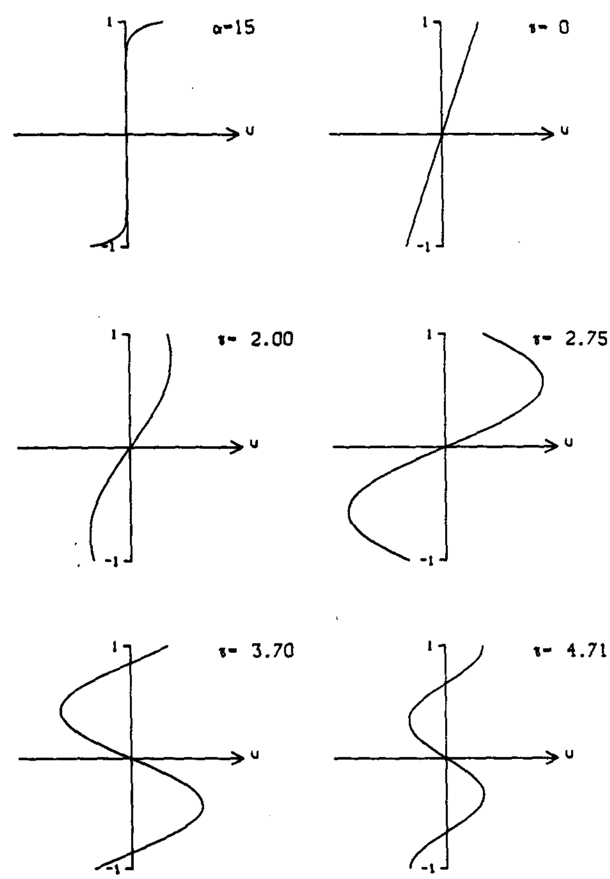


FIG. 5. Velocity profiles corresponding to various points on Fig. 4. The bottom two profiles are on the upper, cusped branch.

(a) $\epsilon \neq 0$, $U = 0$ and $\gamma$ is a continuous variable
(b) $\gamma \to \pi$ with $U_0 \equiv U/\sin\gamma$ fixed.

The solid line in Fig. 6 is the enstrophy–energy relation in case (a). The lowest point is $\delta = \gamma^2 = 0$. The left-hand portion, with the vertical asymptote ($Z/M^2 \to \infty$ as $E/M^2 \to 1/4$), has $\delta = \alpha^2 > 0$. The right-hand section, for which $Z/M^2 \sim \gamma_*^2(E/M^2) + O(E/M^2)^{1/2}$, has $0 > \delta = -\gamma^2 > -\gamma_*^2$. Bifurcating from this curve at $\gamma = \pi$ is the family in (b). For this family the velocity profile in (3.5a) can be rewritten as

$$u = \frac{1}{2}M\{1 + \cos\pi y\} + [2E - 3M^2/4]^{1/2}\sin\pi y \qquad (4.8)$$

and the enstrophy–energy relation of the above is

$$Z/M^2 = \pi^2(E/M^2) - (\pi^2/4). \qquad (4.9)$$

If the inequality (4.5) is satisfied, then the nonsymmetric mode bifurcates first (i.e., at $\gamma = \gamma_{11} < \pi$) and is the minimum enstrophy solution when $E$ exceeds $\bar{E}(\gamma_{11})$. On the other hand, if $\gamma_{11} > \pi$, then the minimum enstrophy solution is always symmetric. When $E < 3M^2/8$ it is of type (a), while if $E > 3M^2/8$ it is of type (b) and the velocity profile is given by (4.8).

*d. The general case: $U \neq 0$ and $M \neq 0$*

Now consider the general case in which the circulation and the momentum are both nonzero. Some

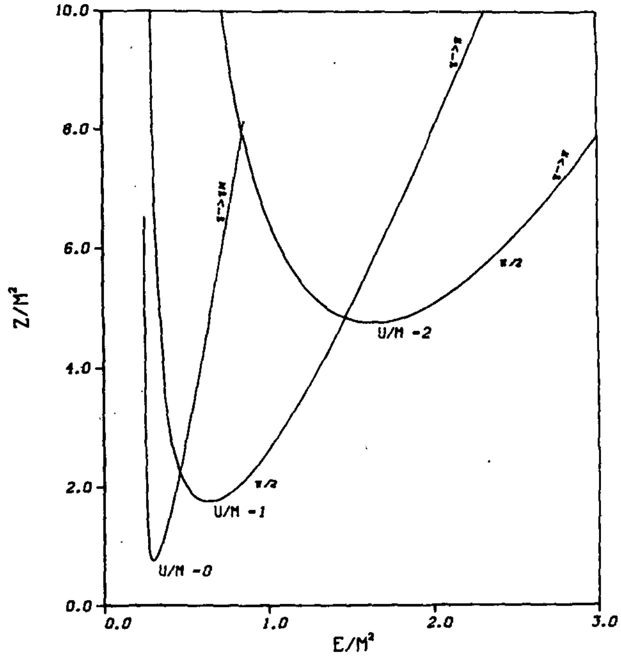


FIG. 7. Various energy–enstrophy relations when $U$ and $M$ are both nonzero. The points at which $\gamma = \pi/2$ are indicated on the curves $U/M = 1$ and $U/M = 2$. When $U/M = 0$ the point $\gamma = \pi/2$ is very close to the minimum of Fig. 6.

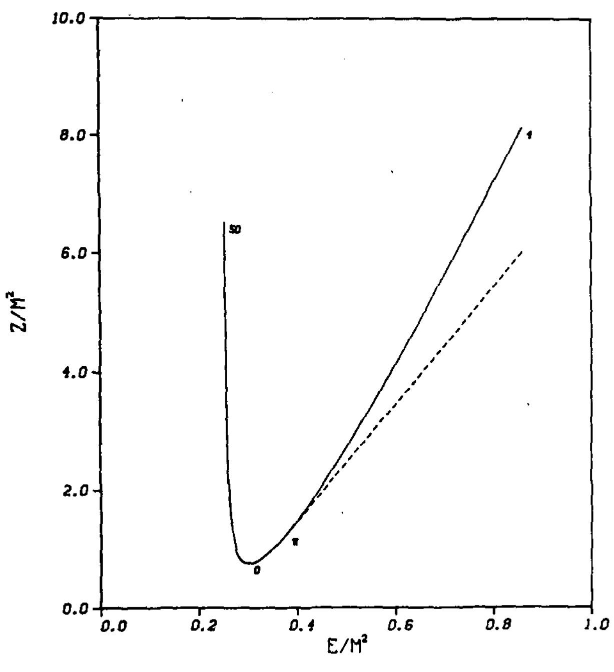


FIG. 6. The solid curve shows the energy enstrophy relation of the symmetric solutions in (3.5) with $U = 0$ and $2500 > \delta > -16$. With this restriction on the range of $\delta$, only a portion of the relation is constructed above. Selected values of $\gamma$ or $\alpha$ are shown on the plot. The dashed straight line is the energy–enstrophy relation of the symmetric profile in (4.8) and (4.9).

typical symmetric enstrophy–energy relations are shown in Fig. 7. Also redrawn on this expanded horizontal scale is the solid curve from Fig. 6, i.e., the symmetric enstrophy–energy relation when $U = 0$. It seems plausible that as $U/M$ is reduced and becomes much less than one, the solid curves in Fig. 7 must somehow continuously deform into the dashed straight line in Fig. 6. This transition is illustrated in Fig. 8, which shows the symmetric enstrophy–energy relations when $U/M = 0$ (the solid curve) and $U/M = 1/8$ (the dashed curve). The lowest branch of the dashed curve is closely approximated by the minimum enstrophy solution in Fig. 6. Thus, as $E/M^2$ increases, the *symmetric* solution with the smallest enstrophy for given energy, momentum and circulation increasingly resembles (4.8) and (4.9). However, as in the discussion which follows those equations, if the channel is longer than 2.31, then the nonsymmetric solution bifurcates from the symmetric relation in Fig. 7 and provides a solution with even smaller enstrophy. In most atmospheric applications the inequality (4.5) will be effortlessly satisfied. Consequently, when $E > \bar{E}(\gamma_{11})$, the minimum enstrophy solution is a finite-amplitude Rossby wave rather than a parallel flow.

*e. Are the stationary solutions extrema?*

The solutions of (3.2) are stationary points of (3.1). They may be maxima, saddle-points or minima. A little

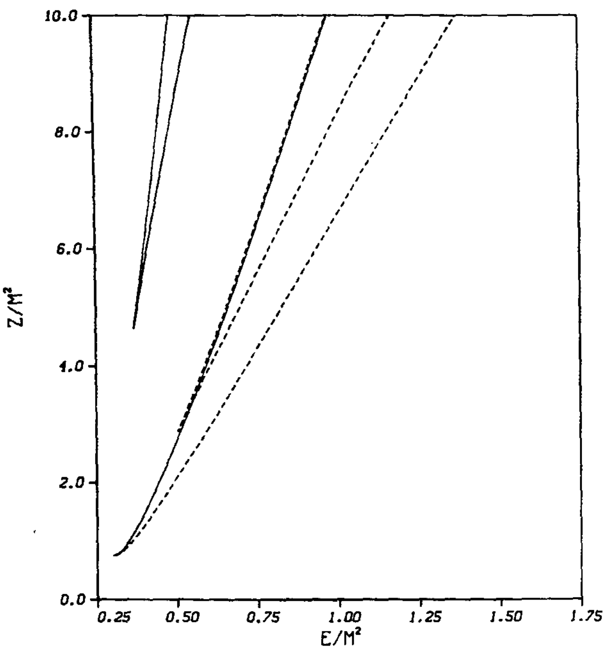


FIG. 8. The solid curve is the right-hand portion ($\delta < 0$) of energy–enstrophy relation of the symmetric profiles when $U/M = 0$. This same relation is shown in Figs. 6 and 7. The dashed curve the energy–enstrophy relation of symmetric profiles when $U/M = 1/8$.

thought shows they cannot be maxima: by simply adding very small-scale vorticity to any flow it is possible to increase the enstrophy without altering the energy, momentum or circulation.

The most straightforward means of deciding between saddle-points and minima is an examination of the second variation of $F$. Thus, if

$$\psi = \psi_s + \eta \tag{4.10}$$

where $\psi_s$ is now the stationary point obtained from (3.2), then

$$\delta^2 F = \int \frac{1}{2}(\nabla^2\eta)^2 + \delta\frac{1}{2}\nabla\eta\cdot\nabla\eta dA. \tag{4.11}$$

If $\delta = \alpha^2 \geqslant 0$, then the second variation is manifestly positive and the stationary point is a minimum.

In fact, if $-\delta \leqslant \min(\pi^2, \gamma_{11}{}^2)$, then $\delta^2F$ is again positive. One proves this by minimizing the homogeneous functional

$$G[\eta] = \int (\nabla^2\eta)^2 dA \Big/ \int \nabla\eta\cdot\nabla\eta dA \tag{4.12}$$

where $\eta_x$ is zero at $y = \pm 1$ and $\eta(x + a, y) = \eta(x, y)$. The smallest value of $G$ turns out to be the smaller of $\pi^2$ and $\gamma_{11}{}^2$. In the first case we conclude that the stationary, symmetric solution constructed earlier is in fact a minimum. In the second case we know that the stationary symmetric solution is a minimum up to the point at which the nonsymmetric solution bifurcates (i.e., $\gamma = \gamma_{11}$).

After the bifurcation to the nonsymmetric solution $\gamma$ is "frozen" at $\gamma_{11}$ and the second variation is not positive definite. In fact, it is easy to construct a perturbation for which it is zero: merely shift the phase of the nonsymmetric part, $\psi'$. Thus, although $\delta^2F$ is not positive, it is nonnegative because the minimum value of $G$ in (4.12) is $\gamma_{11}{}^2$. Thus, apart from trivial alterations in phase, which leave the enstrophy unaltered, the nonsymmetric solutions with $\gamma = \gamma_{11} < \pi$ are minima.

## 5. Conclusions

It has been shown that if one specifies the energy, momentum and circulation of a quasi-geostrophic fluid in a $\beta$-plane channel then there is a minimum enstrophy implied by these constraints. The form of the minimum enstrophy solution depends on the aspect ratio of the channel. If the zonal length of the channel is less than 1.15 times its meridional width, then the solution which minimizes the enstrophy is always a parallel flow. However, in the geophysically relevant case when this inequality is reversed, the form of the solution is decided by the ratio of the energy to the square of the momentum: $e = E/M^2$. When this parameter is below a critical value the minimum enstrophy solution is again a parallel flow. When this critical value is exceeded, the minimum enstrophy solution is a finite-amplitude Rossby wave.

It is easy to physically interpret and summarize these results. Imagine fixing the momentum and increasing the energy of the fluid. A plausible candidate for a minimum enstrophy solution is a parallel flow which increasingly resembles that in Fig. 2b as $e \equiv E/M^2$ grows. This is essentially the solution in (4.8) and (4.9), and intuitively one is storing energy in the term proportional to $\sin\pi y$ without increasing the momentum. However, an alternative is a wavy solution in which the energy is stored in the asymmetric component. Once again this can be done without increasing the momentum. Not surprisingly, the flow with the smallest enstrophy is just the one with the smallest wavenumber. This is the asymmetric solution if $\gamma_{11} < \pi$, or equivalently, $a > 2.30$.

For an ideal fluid the stability of the minimum enstrophy solutions is guaranteed by the variational construction. All adjacent flows with the same energy and momentum have more enstrophy, and consequently are inaccessible. The stability of the stationary, but nonextremal solutions is also of interest and some preliminary results are in Barsugli (1986).

In a nonideal fluid the heuristic "selective decay principle" suggests that the minimum enstrophy solutions are not only stable, but attract arbitrary initial conditions because scale-selective dissipation preferentially destroys enstrophy. This is not a rigorous deduction of the preceding calculation and testing it requires numerical computations beyond the scope of the present work. However, even if the selective decay principle is invalidated by these calculations, the variational problem should still provide a useful framework for interpreting and diagnosing them. For instance, one can visualize the evolution of fluid in a channel as a moving point on Figs. 4 or 6. The point must lie above the minimum enstrophy curve and its orbit in the ($z$, $e$) plane might be useful means of summarizing a numerical experiment.

It must be admitted that Hou and Farrell's (1986) experiments do not generally support the selective decay hypothesis. This is because they used a "square" channel ($a$ = 2). With this modest, and perhaps unrealistic, aspect ratio the minimum enstrophy solution is always a parallel flow. Thus the persistent Rossby wave which develops in their calculation is not a minimum enstrophy solution. It may be that their integration was stopped before the excess enstrophy was destroyed by scale-selective dissipation. It would be of interest to compare extended integrations in channels with disparate zonal lengths.

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## APPENDIX

### Algebraic Details of the Symmetric Solutions

From (3.5) one has the following expression for $E$, $M$ and $Z$ when the solution is symmetric (independent of $x$):

$$\left.\begin{aligned}\bar{E}(\gamma,\epsilon,U)&=(\epsilon/\gamma^2)^2\left\{1+\frac{1}{2}\sec^2\gamma-\frac{3}{2}\gamma^{-1}\tan\gamma\right\}\\&\quad+(U^2/2)\{\mathrm{cosec}^2\gamma-\gamma^{-1}\cot\gamma\}\\\bar{M}(\gamma,\epsilon,U)&=2(\epsilon/\gamma^2)\{1-\gamma^{-1}\tan\gamma\}\\\bar{Z}(\gamma,\epsilon,U)&=(\epsilon/\gamma^2)^2(\gamma^2/2)\{\sec^2\gamma-\gamma^{-1}\tan\gamma\}\\&\quad+(U^2\gamma^2/2)\{\mathrm{cosec}^2\gamma+\gamma^{-1}\cot\gamma\}\end{aligned}\right\}\tag{A1}$$

if $\delta = -\gamma^2 < 0$.

If $\delta = \alpha^2 > 0$, then

$$\left.\begin{aligned}\bar{E}(\alpha,\epsilon,U)&=(\epsilon/\alpha^2)^2\left\{1+\frac{1}{2}\mathrm{sech}^2\alpha-\frac{3}{2}\alpha^{-1}\tanh\alpha\right\}\\&\quad+(U^2/2)\{-\mathrm{cosech}^2\alpha+\alpha^{-1}\coth\alpha\}\\\bar{M}(\alpha,\epsilon,U)&=2(\epsilon/\alpha^2)\{1-\alpha^{-1}\tanh\alpha\}\\\bar{Z}(\alpha,\epsilon,U)&=(\epsilon/\alpha^2)^2(\alpha^2/2)\{-\mathrm{sech}^2\alpha+\alpha^{-1}\tanh\alpha\}\\&\quad+(U^2\alpha^2/2)\{\mathrm{cosec}^2\alpha+\alpha^{-1}\coth\alpha\}\end{aligned}\right\}.\tag{A2}$$

The overbar in the above definitions emphasizes that these are the energy, momentum and enstrophy of *symmetric* solutions. The final step of the variational calculation is to solve for $\delta$ and $\epsilon$ as functions of $E$, $M$ and $U$. One then expresses $Z$ as function $E$, $M$ and $U$ using (A1c) and (A2c). This has been done numerically and the results are shown in several figures. If $M$ is nonzero it is convenient to use

$$e\equiv E/M^2,\quad z\equiv Z/M^2,\, U/M\tag{A3}$$

as independent variables. It is clear from (A1) and (A2) that $z$ is a function of $U/M$ and $e$ alone, so that the definitions in (A3) reduce the number of independent variables from three ($E$, $M$, $U$) to two ($e$, $U/M$). The case $M$ = 0 requires a separate discussion.

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