Rates, pathways, and end states of nonlinear evolution in decaying two-dimensional turbulence: Scaling theory versus selective decay

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A recently proposed scaling theory of two-dimensional turbulent decay, based on the evolutionary pathway of successive mergers of coherent vortices, is used to predict the rate and end state of the evolution. These predictions differ from those based on the selective-decay hypothesis and traditional ideas of spectrum evolution, and they are in substantially better agreement with numerical solutions at large Reynolds number.

The spectrum evolution of decaying two-dimensional turbulence at large Reynolds number shows movement of energy toward larger scales and dissipation of vorticity variance (enstrophy) at smaller scales. However, its evolution is also characterized by the emergence of and dynamical control by coherent vortices. The mechanism known as selective decay has recently been reiterated as apt for this system.[1] In this Brief Communication, we present an assessment of this hypothesis on the basis of the scaling theory we recently proposed,[2] as well as two high-resolution numerical solutions.[1,3] In particular, we examine (1) the pathway and rate by which the system approaches the dipole-vortex flow configuration that marks the cessation of significant nonlinear evolution and (2) the relation between that final dipole and the flow configuration predicted by the selective-decay hypothesis.

We find that scaling theory predicts a longer time to reach the final dipole than classical estimates of spectrum evolution and that the final dipole has a larger enstrophy, larger maximum vorticity, and smaller radius than that predicted by selective decay. Furthermore, these distinctions are supported by the numerical solutions.

Inviscid two-dimensional flow conserves both energy $\mathscr{E}$ and enstrophy Z, where

$$\mathscr{E}\equiv\frac{1}{(2\pi L)^2}\int\frac{1}{2}|\mathbf{u}|^2\,d\mathbf{x},\quad Z\equiv\frac{1}{(2\pi L)^2}\int\frac{1}{2}\zeta^2\,d\mathbf{x},\tag{1}$$

and $\zeta=\hat{\mathbf{z}}\cdot\nabla\times\mathbf{u}$ is the vorticity. [The domain is a doubly periodic one in (x,y) over a distance $2\pi L$, and $\hat{\mathbf{z}}$ is the unit vector normal to the plane of motion.] As a consequence any nonlinear evolution that tends to broaden the wave-number spectra will move the centroids of the energy and enstrophy spectra to larger and smaller scales, respectively.[4] Because viscosity is preferentially dissipative on small scales, it is plausible that enstrophy should be dissipated more rapidly than energy. This argument motivated an early proposal[5] for self-similar spectrum evolution in which $\mathscr{E}$ is conserved and is the only property of the initial conditions affecting the long-term evolution; hence

$$Z\sim t^{-2}.\tag{2}$$

A traditional view of the evolutionary pathway is that the peak wave number of the energy spectrum, k_*, proceeds by geometric progression on a circulation time scale, $t_c(t)$; i.e., $dk_*/dt\sim-k_*/t_c$, where

$$t_c(t)=Z^{-1/2}(t)=k_*^{-1}(t)\mathscr{E}^{-1/2}.\tag{3}$$

This traditional view also implies (2).

The idea that enstrophy is dissipated more rapidly than energy also motivated the selective-decay hypothesis,[6,7] viz., that the evolution proceeds to an extremal state in which Z is minimal for a given (initial) $\mathscr{E}$. Once the flow reaches the extremal state, there is no further nonlinear evolution, and a small viscosity induces only a slow diffusive decay. The configuration for this state is any superposition, translation, or 90° rotation of parallel flow in the gravest wave number, $\cos(x/L)$. The superposition corresponding to a dipole is

$$\psi(x,y,t)=\sqrt{2\mathscr{E}L^2}[\cos(x/L)-\cos(y/L)],\tag{4}$$

where ψ is the streamfunction, $\mathbf{u}=\hat{\mathbf{z}}\times\nabla\psi$, and $\zeta=\nabla^2\psi\,[=-\psi/L^2$ for (4)]. This flow has the properties

$$\zeta_{\text{ext}}=2^{3/2}\frac{\sqrt{\mathscr{E}}}{L},\quad Z=\frac{\mathscr{E}}{L^2},\quad a=\frac{\sqrt{2}\pi}{3}L,\tag{5}$$

where ζ_{ext} is the vorticity extremum, and a, the radius of the individual vortices in the dipole, is defined as the distance along a line connecting the vortex centers at which ζ decays to, say $\zeta_{\text{ext}}/2$. Combining the selective-decay value of the final enstrophy (5) with the decay law (2) gives an estimate for the nonlinear final time t_f for the arrival of the solution at the extremal state (4):

$$t_f\sim L/\sqrt{\mathscr{E}}.\tag{6}$$

Note that L and $\mathscr{E}$ are the only dimensional parameters in these results.

It is evident from the numerical solutions that the principal pathway for spectrum evolution, once the vortices have become dominant, is merger of pairs of like-sign vortices. Merger occurs when the essentially inviscid and chaotic mutual advection among all vortices brings a pair close enough together. The average peak vorticity of the vortices is relatively constant throughout the evolution.[3] Over a sufficiently long time, the chaotic advection will effect all possible mergers, and when there remain two vortices, one of each sign, the advection becomes regular and the evolution ceases to be significantly nonlinear.[17] A vortex di-

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pole is a uniformly propagating stationary state of the inviscid dynamics, called a modon in some contexts.[8,9]

We now contrast the predictions in (4)–(6) with those of scaling theory,[2] which asserts that there is a second conserved quantity (in addition to $\mathscr{E}$) affecting the long-term evolution: the average peak vorticity inside the vortex cores. We denote this extremal value of ζ by $\zeta_{\rm ext}$. The emergence of coherent vortices from random initial conditions occurs on a time scale $t_0 \sim 1/\zeta_{\rm ext} \sim t_c(0)$. Thereafter, the evolution of the spectrum is considerably slower than in the traditional view stated above, and thus one can usefully speak of an arrest or suppression of the turbulent cascade by the coherent vortices;[10,11] this language is not intended to imply a total cessation of nonlinear evolution, as some have characterized it.[1] Under conditions where the vortex population attributes are not broadly distributed, this stage of evolution exhibits scaling behavior in which both the means of these attributes (e.g., number N, peak amplitude $\zeta_{\rm ext}$, and radius a) and the bulk statistical moments (e.g., $\mathscr{E}$ and Z) vary algebraically with time.[2,3] Specifically,

$$N(t) = N(t_0)(t/t_0)^{-\xi}, \quad Z(t) = Z(t_0)(t/t_0)^{-\xi/2},$$

$$a(t) = a(t_0)(t/t_0)^{\xi/4}, \quad \zeta_{\rm ext}(t) = \zeta_{\rm ext}(t_0),$$

$$\mathscr{E}(t) = \mathscr{E}(t_0). \tag{7}$$

The scaling relations result from associating the statistical moments with the vortex attributes:

$$\mathscr{E} \sim N\zeta_{\rm ext}^2 a^4/L^2, \quad Z \sim N\zeta_{\rm ext}^2 a^2/L^2. \tag{8}$$

The exponent ξ has been found in both fluid and modified-point-vortex solutions to be $\xi \approx 0.70$–0.75.[2,3] Note that the enstrophy decay rate is much slower here than estimated by (2): i.e., $\xi/2 \approx 0.37 < 2.0$.

The scaling theory was originally proposed to describe a turbulent fluid containing a large number of well-separated coherent vortices, and this is the regime in which it has been confirmed.[2,18] We now extrapolate the theory to small numbers of vortices, estimating the time t_f required to reach the final dipole state by setting $N(t_f) = 2$ in (7). The emergent vortices have size and spacing that can be related to the wave-number spectrum of the random initial conditions: they are reasonably closely packed, $a(t_0) \sim k_*^{-1}(0)$, and their spacing corresponds to the distance between neighboring extrema, $L/N^{1/2}(t_0) \sim k_*^{-1}(0)$. Thus, $N(t_0)a(t_0)^2 \sim L^2$. Using (7) and (8) then gives

$$t_f \sim \zeta_{\rm ext}^{-1}(\zeta_{\rm ext}L/\sqrt{\mathscr{E}})^{2/\xi}. \tag{9}$$

Compare this to the selective-decay estimate, (6), rewritten as $t_f = \zeta_{\rm ext}^{-1}(\zeta_{\rm ext}L/\sqrt{\mathscr{E}})^1$: The exponent here is larger, $2/\xi \approx 2.76 > 1$, indicating a much longer time to reach the final dipole. In the same manner, we can use (7)–(9) to estimate the peak vorticity, enstrophy, and radius of the vortices in the final dipole:

$$\zeta_{\rm ext} \sim N_0^{1/2}(\sqrt{\mathscr{E}}/L), \quad Z \sim N_0^{1/2}(\mathscr{E}/L^2), \quad a \sim N_0^{-1/4}L. \tag{10}$$

A comparison of (5) and (10) shows that scaling-theory estimates of the properties of the final dipole differ significantly from selective-decay estimates when the initial vortex population is large.

We have observed the inexorable but lengthy evolution to the final dipole in several low-resolution numerical solutions (128^2), and one example at higher resolution (512^2) and Reynolds number has recently been reported.[1] These dipoles have a qualitative similarity to (4) in that there are two vorticity extrema of opposite sign, separated by a distance of $O(L)$, and there is essentially no further nonlinear evolution. However, the vorticity amplitudes and enstrophies are appreciably larger, and the radii of the vortices are appreciably smaller than in (5). Also, the rates of enstrophy decay are slower than (2), and the final times t_f are appreciably larger than (6).

The final dipole which emerges is a member of a family of dipoles with continuously varying orientation and separation, whose dynamics consists of relatively simple propagation. Considering that the pathway to the final dipole is a chaotic sequence of mergers of close same-sign vortices, there is no reason for a particular configuration to be selected. Indeed, although the final dipole found in Ref. 1 is very close to the orientation and separation in the stationary, selective-decay solution (4), some low-resolution solutions exhibit final dipoles with different orientations and separations. This range of final dipole shapes and configurations also contradicts the alternative statistical–mechanical proposal that the final configuration is given by the sinh-Poisson equation.[12]

In solutions with a broadband random initial energy spectrum peaked at $k_* = O(L^{-1})$,[1,13] there are relatively few large vortices in the emergent vortex population, along with many more small vortices. These large vortices dominate the chaotic mutual advection and relatively few mergers among them are required to arrive at the final dipole. Thus, the associated normalized vorticity amplitude and enstrophy will have only modest excesses above the selective-decay predictions (5). The relevance of our temporal scaling theory to broadband solutions is not yet clear (although some degree of spatial scaling behavior is observed[14]). For example, the enstrophy decay (Fig. 1 in Ref. 1) does not have a uniform power-law form, although, as pointed out by the authors, its rate is substantially slower than the traditional estimate (2); however, after $t \approx 30$, its decay exponent does appear to be close to $-\xi/2$, as in (7).

Scaling theory is clearly apt for solutions with initially narrow-band spectra, in which the emergent vortex population attributes are not very broadly distributed. The solution in Ref. 3, which is in this class, has been integrated to $t = 150$, corresponding to approximately 1000 circulation times. This integration time is comparable to that in Ref. 1, but the larger $k_*(0)$ (and hence $\zeta_{\rm ext}$ and N_0) makes it impractical to run out to the final dipole; from (9), $t_f \approx 3500$, which is strikingly larger than the traditional prediction (6), $t_f \approx 1$. In addition, at present numerical resolutions, the influence of diffusion over $\Delta t = 3500$ would cause appreciable departures from the (infinite Rey-

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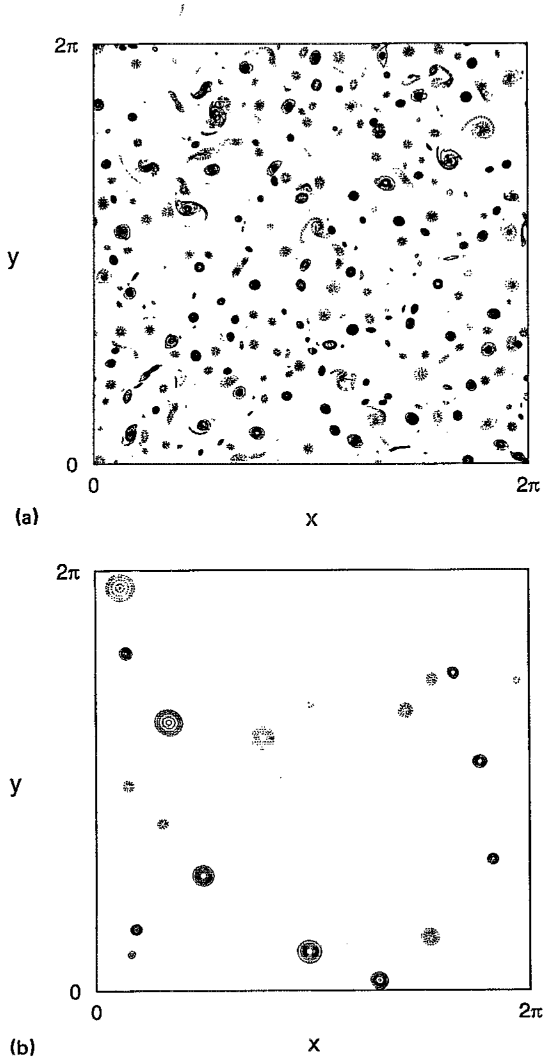

FIG. 1. $\zeta(x,y)$ at (a) $t=5$ and (b) $t=150$. The contours are ± 12, ± 24,..., and the negative contours are dashed.

nolds number) scaling theory. The vorticity fields at $t=5$ (with $N=223$) and at the latest time, $t=150$ (with $N=19$), are shown in Fig. 1. It is clear from this figure that the vortices in the final dipole will be very much smaller and stronger than the selective-decay predictions (5), in accord with scaling theory (10).

Recently, some new statistical mechanical arguments have been proposed for two-dimensional flow.[15,16] Extremal hypotheses about the evolution of nonlinear dynamical systems, such as these and the selective-decay hypothesis, can in some instances be accurate in determining the end states of evolution. However, by their silence about dynamical pathways, they can also be misleading if the associated rates become very slow, or even zero if there does not exist an evolutionary pathway between a particular initial state and the predicted extremal state. By incorporating the pathway for two-dimensional turbulent decay, scaling theory provides new predictions for the rate and end state which differ from those of previous hypotheses and which are in substantially better agreement with numerical solutions at large Reynolds number.

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[17]There may be configurations of more than two vortices whose trajectories have no approaches close enough to initiate a merger (e.g., there are periodic trajectories without close approaches for three point vortices in an open domain). However, as yet no such end states of the chaotic, nonlinear evolution have been found, hence we assess this possibility as unlikely at best.

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