J. Fluid Mech. (1980), *vol.* 99, *part* 4, *pp.* 715–724
Printed in Great Britain

On wave–current interaction theories of Langmuir circulations

By S. LEIBOVICH

Sibley School of Mechanical and Aerospace Engineering,
Cornell University, Ithaca, N.Y. 14853

(Received 15 May 1979 and in revised form 21 December 1979)

The Craik–Leibovich (CL) equations for Langmuir circulations are shown to be an Eulerian approximation to an exact theory of the generalized Lagrangian mean (GLM) due to Andrews and McIntyre. Derivation of the CL equations using the GLM formalism is decisively simpler than the original method. The CL theory is then compared to other wave–current interaction theories of Langmuir circulations, notably those of Garrett and of Moen.

1. Introduction

Craik & Leibovich (1976; designated as CL hereafter) and Leibovich (1977*a*) have presented a theory of Langmuir circulation that appears to be consistent with the facts known about the occurrence of Langmuir circulations in oceans and lakes, and with available laboratory experiments (Faller 1978).

The theory rests upon a set of equations, set out in their fullest form in Leibovich (1977*b*), for the Eulerian-mean flow in surface layers under the influence of nearly irrotational surface waves, and a wind stress which creates rotational currents that are weak compared to particle speeds in the waves. These equations, in which wave activity is represented as a rectified effect, predict convective activity similar to Langmuir circulations to result from either of two mechanisms. One mechanism (explored in CL, Leibovich 1977*a*, and Leibovich & Radhakrishnan 1977) requires a surface wave field with a high degree of spatial structure, which can, through the CL equations, act to directly force circulatory motion. The second mechanism (first described by Craik 1977 and further explored by Leibovich 1977*b*) derives circulations from the CL equations as an inviscid instability of a unidirectional current in the presence of a wave field without special spatial structure.

We refer to CL, to Leibovich (1977*a*) to Leibovich & Radhakrishnan (1977), to Craik (1977), and to Leibovich (1977*b*) for a discussion of the observed phenomenon of Langmuir circulation and for the development of the theory, and we refer to any results obtainable from the CL equations as a 'CL theory'.

Another wave–current interaction theory for Langmuir circulations has been proposed by Garrett (1976) that contemplates a situation which is, in part at least, similar to that postulated in the CL theories, viz., nearly irrotational waves and a weaker rotational current. A variant of Garrett's theory is given by Moen (1978). In these approaches, in contrast to CL, refraction of trains of surface waves by horizontal shear in the Langmuir circulations plays a dominant role. The refraction results from

0022-1120/80/4583-1790 $02.00

the presence of the relatively weak rotational mean current, and is computed by kinematic wave (WKB) theory. Garrett has to invoke an additional *ad hoc* assumption concerning dissipation of wave energy; Moen does not need this but his theory also has eclectic aspects. In § 3, we discuss these two theories, and their connections and contrasts with the CL theories.

In the averaging procedure used by CL, a correction of the wave field to account for the rotational current is also required. This correction need not be computed explicitly, however. All that is needed is the vorticity vector associated with the wave activity; or, expressed another way, the vorticity fluctuations induced by the waves. In the final CL equations, the only vestige of the surface waves lies in a 'vortex force' term containing the Stokes drift due to the irrotational, lowest-order, wave field. One purpose of the present note is to point out that any rational wave–current interaction theory for Langmuir circulations that shares the CL hypotheses (that a small amplitude surface wave field exists and is, to a good approximation, irrotational) must be represented by the CL equations. This statement follows from the fact that the CL equations are derived from the Navier–Stokes equations by rational methods in which the errors committed can be estimated systematically. In this paper, we show that the CL equations are the appropriate approximation to the exact wave/mean flow interaction equations presented by Andrews & McIntyre (1978, designated AM hereafter). In particular, if the generalized Lagrangian-mean (GLM) equations of the AM theory are specialized to homogeneous fluids subjected to any small-amplitude surface wave fluctuation dominated by its irrotational part, they reduce to the CL theory. The present derivation allows one to see the errors incurred in the CL equations from another perspective since the process of approximation is done without the added distractions of averaging.

This derivation from the AM theory is of interest for other reasons. First, the alternative derivation is remarkably simple. Only the leading terms in the implied perturbation expansions need now be considered. In particular, the vorticity fluctuations induced by the waves do not enter explicitly at all. A similarly drastic simplification is found when the Coriolis effects considered by Hasselmann (1970) and Huang (1979) are taken into account. Second, the original derivation of the CL equations by perturbation expansions required a specific ordering of effects; exactly the same ordering is, not surprisingly, necessary to obtain the CL equations from the GLM equations.

Relaxation of the CL ordering assumptions seems to preclude the derivation of a closed set of rectified equations by the method of Craik & Leibovich (1976). For example, the method cannot be applied if the ordering originally assumed by Craik (1970, in the paper containing the seeds of what eventually evolved into the CL theory) is adopted. For this reason, the introduction by Leibovich & Ulrich (1972) of the ordering incorporated in the CL theories – an ordering generally in accord with observed motions involving wind-generated waves and currents in the ocean – was an essential step in the derivation of equations describing nonlinear rectified effects of waves on currents by the CL method. By contrast, the GLM equations of AM describe such rectified effects without the need to invoke the CL assumptions, and suggest that some of the essential mathematical structure underlying the existence of Langmuir-circulation instabilities might carry over into conditions under which the CL equations do not apply: in particular, when wave orbital speeds are not large compared to mean currents (so the waves can no longer be essentially irrotational), or when finite wave amplitudes must be accounted for. That the existence of wave-driven convective motions in the former case is likely was indeed indicated by the linearized analysis of Craik (1970) (in which the waves were assumed infinitesimal but rotational).

2. Reduction of AM to CL

We begin with the mean-flow evolution equation for momentum in the GLM (generalized Lagrangian mean) description as presented by Andrews & McIntyre (1978) (their equation 3.8):

$$\bar{D}_i^{\mathrm{L}}(\bar{u}_i^{\mathrm{L}} - p_i) + \bar{u}_{k,i}^{\mathrm{L}}(\bar{u}_k^{\mathrm{L}} - p_k) + 2(\mathbf{\Omega} \times \bar{\mathbf{u}}^{\mathrm{L}})_i + \pi_{,i} = \nu\{\overline{(\nabla^2 u_i)}^{\mathrm{L}} + \overline{\xi_{j,i}(\nabla^2 u_j)^l}\}, \tag{1}$$

where the equations have been written for a constant density fluid in laminar motion, or in turbulent motion parameterized by a constant eddy viscosity. Here

$$\pi = \frac{1}{\rho}\bar{P}^{\mathrm{L}} - \tfrac{1}{2}\overline{\mathbf{u}^{\xi}.\mathbf{u}^{\xi}} + \overline{(\mathbf{\Omega} \times \boldsymbol{\xi}).\mathbf{u}^{\xi}},$$

where P is the fluid pressure; the notation is essentially that used by AM, and the meaning of the other terms will be discussed below. In (1), $\mathbf{\Omega}$ is the angular velocity of a rotating reference frame: the CL theory set $\mathbf{\Omega} = 0$, and its inclusion is not necessary for our present purposes. We have introduced it here temporarily because the joint development of an Ekman layer and Langmuir circulations can presumably be described by this set of equations (and by the CL set for $\mathbf{\Omega} \neq 0$).

To define the quantities appearing in (1), we briefly restate as much of the AM formulation as required here; the reader is referred to AM for the details of their theory. Consider two sets of trajectories with Lagrangian velocities

$$\frac{d\mathbf{x}}{dt} = \bar{\mathbf{u}}^{\mathrm{L}}$$

and

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}, t)$$

respectively, where $\mathbf{u}$ is the actual instantaneous velocity of a fluid particle and $\bar{\mathbf{u}}^{\mathrm{L}}$ is the corresponding quantity for a fictitious, or reference, fluid motion that takes place with a 'Lagrangian-mean velocity'. An oscillatory displacement, described by the vector field $\boldsymbol{\xi}(\mathbf{x}, t)$, from the reference path is assumed to take place, and the two sets of trajectories are related by the map

$$\mathbf{x} \mapsto \mathbf{X}(\mathbf{x}, t) \equiv \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t).$$

Let an overbar $\overline{(\ \ )}$ refer to any suitable averaging operation that may be contemplated in an Eulerian framework. Then the generalized Lagrangian-mean operator $\overline{(\ \ )}^{\mathrm{L}}$ applied to a tensor field (say ϕ) of any rank is defined in terms of $\overline{(\ \ )}$ by

$$\overline{\phi(\mathbf{x}, t)}^{\mathrm{L}} = \overline{\phi(\mathbf{X}(\mathbf{x}, t), t)}.$$

In AM, the notation

$$\phi^{\xi}(\mathbf{x}, t) = \phi(\mathbf{X}(\mathbf{x}, t), t)$$

is used, and ϕ^l is defined to be

$$\phi^l = \phi^\xi - \bar{\phi}^L.$$

Thus the 'Lagrangian-mean velocity' of the field associated with the instantaneous velocity field $\mathbf{u}(\mathbf{x},t)$ is defined to be

$$\bar{\mathbf{u}}^L(\mathbf{x},t) = \overline{\mathbf{u}^\xi}.$$

The 'Lagrangian-mean material derivative', $\bar{D}^L$, is defined to be

$$\bar{D}^L(\quad) = (\quad)_{,t} + \bar{\mathbf{u}}^L.\nabla(\quad) \tag{2}$$

and

$$\bar{D}^L\boldsymbol{\xi} = \mathbf{u}^l. \tag{3}$$

Furthermore, the vector $\boldsymbol{p}$ is identified in Andrews & McIntyre (1978) as the wave pseudomomentum per unit mass, and is defined to be

$$p_i(\mathbf{x},t) = -\overline{\xi_{j,i}[u_j^l + (\boldsymbol{\Omega}\times\boldsymbol{\xi})_j]}. \tag{4}$$

This serves to identify all of the terms in equation (1). As AM emphasizes, (1) is an exact equation, although practical use of it has been restricted to waves of small amplitude, measured by a dimensionless amplitude parameter ϵ, so that the displacements $\boldsymbol{\xi}$ are $O(\epsilon)$ compared to wavelengths, say λ, in the wave field.

We now show that if the motion $\mathbf{u}(\mathbf{x},t)$ is dominated by $O(\epsilon)$ wave activity, and if the waves are irrotational to $O(\epsilon)$ with rotational contributions to the motion $o(\epsilon)$ (typically $O(\epsilon^2)$, as is Stokes wave drift), then the AM equations are equivalent to the CL equations. We note that the latter are equations for the Eulerian mean velocity $\bar{\mathbf{u}}(\mathbf{x},t)$, and not the Lagrangian mean $\bar{\mathbf{u}}^L$. The Eulerian and Lagrangian means are related (AM, equation (2.25)) by

$$\bar{\mathbf{u}}^L = \bar{\mathbf{u}} + \bar{\mathbf{u}}^S \tag{5}$$

where $\bar{\mathbf{u}}^S$ is the (generalized) Stokes correction. Let $\mathscr{U}$ be a typical velocity scale for the mean motion (we may take $\mathscr{U}$ to be the larger of the scales for the Lagrangian or the Eulerian mean motion). For ϵ small, and $|\bar{\mathbf{u}}| = O(\mathscr{U}) = o(|\mathbf{u}'|)$, as postulated in CL (CL specifically assume that $\mathscr{U} = O(|\bar{\mathbf{u}}^S| = O(\epsilon^2 c)$, where c is a typical wave speed for convenience, but $\mathscr{U}/c$ can be regarded as an independent small parameter), AM (p. 619) show that $\bar{\mathbf{u}}^S = \overline{\boldsymbol{\xi}.\nabla\mathbf{u}'}$ (see also Phillips 1966, p. 31), where $\mathbf{u}'$ is the Eulerian wave velocity perturbation, i.e.

$$\mathbf{u}' = \mathbf{u}(\mathbf{x},t) - \bar{\mathbf{u}}(\mathbf{x},t).$$

More specifically, retaining $\mathscr{U}/c$ as an independent small $o(\epsilon)$ parameter, the use of AM (2.27) shows that

$$\bar{\mathbf{u}}^S = \overline{\boldsymbol{\xi}.\nabla\mathbf{u}'} + O(\epsilon^2\mathscr{U}). \tag{6}$$

Furthermore, AM show (p. 631) under the circumstances just stated, with $\boldsymbol{\Omega} = 0$, and assuming that $\overline{(\quad)}$ is an average in time (by explicitly introducing multiple time scales, as in Leibovich 1977*a*, or implicitly by introducing a smoothed running time-averaging operator, as in Bretherton 1971), that $\bar{\mathbf{u}}^S = \boldsymbol{p} + O(\epsilon^3)$. In fact, the error term is smaller. From (2) and (3), $\boldsymbol{\xi}_{,t} = \mathbf{u}^l + O(\epsilon\mathscr{U})$, and $\mathbf{u}^l = \mathbf{u}' + O(\epsilon\mathscr{U})$ (AM 2.28). Furthermore, $\mathbf{u}'$ is assumed to be irrotational at $O(\epsilon)$, and to derive its vorticity from the weak mean current. Thus curl $\mathbf{u}' = O(\epsilon\mathscr{U}/\lambda)$, so $u'_{i,j} = u'_{j,i} + O(\epsilon\mathscr{U}/\lambda)$. But, following AM,

$$\begin{aligned} p_i &= -\overline{u^l_j \xi_{j,i}} = \overline{\xi_j u^l_{j,i}} - \overline{(u^l_j \xi_j)}_{,i} \\ &= \overline{\xi_j u'_{j,i}} - \tfrac{1}{2}\overline{(|\boldsymbol{\xi}|^2)}_{,it} + O(\epsilon^2\mathscr{U}) \\ &= \overline{\xi_j u'_{i,j}} + O(\epsilon^2\mathscr{U}) \end{aligned}$$

or

$$\mathbf{p} = \bar{\mathbf{u}}^{\mathrm{s}} + O(\epsilon^3\mathscr{U}). \tag{7}$$

For currents with $\mathscr{U} = O(\epsilon^2 c)$, the error term is $O(\epsilon^4 c)$. By replacing $\bar{\mathbf{u}}^{\mathrm{L}} - \mathbf{p}$ by $\bar{\mathbf{u}}$ through (5) and (7), equation (1) can be written (for $\boldsymbol{\Omega} = 0$)

$$\bar{u}_{i,t} + \bar{u}^{\mathrm{L}}_j \bar{u}_{i,j} + \bar{u}^{\mathrm{L}}_{k,i}\bar{u}_k + \pi_{,i} = \nu[\overline{(\nabla^2 u_i)}^{\mathrm{L}} + \overline{\xi_{j,i}(\nabla^2 u)^l_j}] + O(\epsilon^2\mathscr{U}^2/\lambda). \tag{8}$$

Assuming no impressed pressure gradients, and small viscous force contributions, the rate of change in time is $O(\bar{u}^{\mathrm{L}}/\lambda)$, where λ is a characteristic length scale. The relative error terms in (8) arising from setting $\mathbf{p} = \bar{\mathbf{u}}^{\mathrm{s}}$ are then $O(\epsilon^2\mathscr{U}/\bar{u})$; if the mean motion itself is second order in ϵ, the terms retained in (8) are correct to $O(\epsilon^4)$.

Noting that

$$\begin{aligned} \bar{u}^{\mathrm{L}}_j \bar{u}_{i,j} + \bar{u}_j \bar{u}^{\mathrm{L}}_{j,i} &= \bar{u}^{\mathrm{L}}_j(\bar{u}_{i,j} - \bar{u}_{j,i}) + (\bar{u}_j \bar{u}^{\mathrm{L}}_j)_{,i} \\ &= \{-\bar{\mathbf{u}}^{\mathrm{L}} \times \operatorname{curl} \bar{\mathbf{u}} + \nabla(\bar{\mathbf{u}} \,.\, \bar{\mathbf{u}}^{\mathrm{L}})\}_i, \end{aligned}$$

equation (8) can be written

$$[\bar{\mathbf{u}}_{,t} + \bar{\mathbf{u}} \,.\, \nabla\bar{\mathbf{u}} + \nabla(\pi + \bar{\mathbf{u}} \,.\, \bar{\mathbf{u}}^{\mathrm{s}})]_i = [\bar{\mathbf{u}}^{\mathrm{s}} \times \operatorname{curl} \bar{\mathbf{u}}]_i + \nu[\overline{(\nabla^2 u_i)}^{\mathrm{L}} + \overline{\xi_{j,i}\nabla^2 u^l_j}].$$

The viscous force term may be reduced to $\nu\nabla^2\bar{\mathbf{u}}$, by the following calculation. Let $\mathbf{F} = \nu\nabla^2\mathbf{u}$, then the viscous contribution to the ith component of the above equation is

$$\overline{F^{\xi}_i} + \overline{\xi_{j,i}F^l_j}.$$

The first term may be simplified since

$$\begin{aligned} \overline{F^{\xi}_i} &= \overline{F_i(\mathbf{x},t) + \boldsymbol{\xi} \,.\, \nabla F_i(\mathbf{x},t)} + O(|\xi|^2\,|F|\lambda^{-2}) \\ &= \nu\{\nabla^2\bar{u}_i + \overline{\boldsymbol{\xi} \,.\, \nabla\nabla^2(\bar{u}_i + u'_i)} + O(\epsilon^2\mathscr{U}/\lambda^2)\}. \end{aligned}$$

This expression can be further reduced, since

$$\overline{\boldsymbol{\xi} \,.\, \nabla\nabla^2(\bar{\mathbf{u}} + \mathbf{u}')} = \bar{\boldsymbol{\xi}} \,.\, \nabla\nabla^2\bar{\mathbf{u}} + \overline{\boldsymbol{\xi} \,.\, \nabla\nabla^2\mathbf{u}'};$$

the first term is identically zero (since $\bar{\boldsymbol{\xi}} = 0$) and the second involves the Eulerian disturbance velocity $\mathbf{u}' = \mathbf{u} - \bar{\mathbf{u}}$, which to $O(\epsilon\mathscr{U})$ is irrotational; thus this second term is of order $\epsilon^2\mathscr{U}$. Consequently,

$$\overline{\mathbf{F}^{\xi}} = \nu\nabla^2\bar{\mathbf{u}} + O\left(\frac{\nu\epsilon^2\mathscr{U}}{\lambda^2}\right).$$

The second part of the viscous force contribution may be approximated in a similar way,

$$\overline{\xi_{j,i}F^l_j} = \nu\overline{\xi_{j,i}[\nabla^2 u'_j + \boldsymbol{\xi} \,.\, \nabla\nabla^2\bar{u}_j]} + O(\nu\epsilon^3\mathscr{U}\lambda^{-2})$$

(using AM (2.28)). Since $\nabla^2\mathbf{u}' = O(\epsilon\mathscr{U}\lambda^{-2})$, and $|\bar{\mathbf{u}}| = O(\mathscr{U})$, each term in the bracket is of the same order, and the entire term is $O(\nu\epsilon^2\mathscr{U}\lambda^{-2})$.

With these approximations, equation (8) can be written solely in terms of the Eulerian mean and the Stokes drift, or

$$\bar{\mathbf{u}}_{,t} + \bar{\mathbf{u}}.\nabla\bar{\mathbf{u}} + \nabla(\pi + \bar{\mathbf{u}}.\bar{\mathbf{u}}^s) = \bar{\mathbf{u}}^s \times \operatorname{curl}\bar{\mathbf{u}} + \nu\nabla^2\bar{\mathbf{u}} \tag{9}$$

with relative error $O(\mathscr{U}/c) + O(\epsilon^2\nu/\mathscr{U}\lambda)$ (that is, this equation times a term

$$[1 + O(\mathscr{U}/c) + O(\epsilon^2\nu/\mathscr{U}\lambda)]$$

holds). Notice that the simplification of the viscous force depends crucially upon the assumption of a constant viscosity coefficient, or constant eddy viscosity. (This was pointed out to me by M. E. McIntyre.)

Equation (9) is the constant-density form of the averaged momentum equations given by Leibovich (1977*b*); they are supplemented by the continuity equation, which requires the Eulerian mean velocity $\bar{\mathbf{u}}$ to be solenoidal.

The arguments that led to the reduction of (1) to the form (9) depend upon the assumption that the waves are dominated by their irrotational part, so that (7) may be invoked. If the reference frame has an angular velocity $\mathbf{\Omega}$, the wave field will have a rotational part, even if viscosity is ignored. If, however, $|\Omega| \ll \sigma$, where σ is a typical wave frequency, the irrotational part of the wave field will continue to dominate the motion, and (7) still applies, but now with $O(\epsilon^2\mathscr{U}) + O(\epsilon^2|\mathbf{\Omega}|\lambda)$ error, or a relative error $O(\mathscr{U}/c) + O(|\mathbf{\Omega}|/\sigma)$. Thus, under the conditions necessary for (9) to hold, and provided $|\mathbf{\Omega}|/\sigma \ll 1$, the Eulerian-mean equations of motion referred to a rotating frame are

$$\bar{\mathbf{u}}_{,t} + \bar{\mathbf{u}}.\nabla\mathbf{u} + 2\mathbf{\Omega}\times(\bar{\mathbf{u}}^s + \bar{\mathbf{u}}) + \nabla(\pi + \bar{\mathbf{u}}.\bar{\mathbf{u}}^s) = \bar{\mathbf{u}}^s \times \operatorname{curl}\bar{\mathbf{u}} + \nu\nabla^2\bar{\mathbf{u}}, \tag{10a}$$

$$\nabla.\bar{\mathbf{u}} = 0. \tag{10b}$$

We note that Huang (1979) derived equations (10) by the CL method, and used them to explore the effects of wave activity on the development of the Ekman layer. (The assumption of horizontal homogeneity ruled out Langmuir circulation effects, although it seems clear that the vertical momentum transport in Langmuir circulations probably modifies the Ekman-layer structure at least as much as does the addition of a Stokes drift contribution to the Coriolis acceleration.) Hasselmann (1970), in a paper on wave forcing of inertial oscillations, derived the inviscid version of Huang's Ekman-layer equation, and thus anticipated his work.

3. Other wave–current interaction theories of Langmuir circulations

Garrett (1976) has suggested an instability mechanism that could produce circulatory motion due to wave–current interactions. The analysis separates the body of water into a thin zone influenced by surface wave activity and the water below, in which wave activity is assumed negligible. In this step, and in the calculations of wave properties, the wavelength of the surface waves is assumed to be small compared to the horizontal scale of features (particularly the surface jet) of convective cells in the Langmuir circulations. If this scale is denoted by L, and the wavenumber by κ, then a basic assumption in the theory is that $(\kappa L)^{-1}$ ($\equiv \hat{\epsilon}$, say) is small.

Adopt a Cartesian (x, y, z) co-ordinate system, with z measured vertically upwards from the mean free water surface. Garrett assumes the existence of an initial current perturbation $(U, 0, 0)$, with a maximum in the spanwise (y) direction. In the upper layer, it is assumed that vertical (z) variations of the current speed can be neglected, or $U = U(y)$; this is consistent with a perturbation procedure valid in the limit $\hat{\epsilon} \to 0$, provided that the depth variation of U is smooth and essentially independent of wave activity. The existence of horizontal shear in U will lead to both refraction and reflexion of surface waves propagating at an angle to the current direction (x). Assuming that the length scale of the current shear is L, the surface waves experience a slow variation of their propagation medium, and Garrett computes their properties approximately using a WKB analysis. This predicts wave refraction (but reflexions are precluded) that leads to an increase of wave energy on the line of maximum U.

Garrett then shows that the average of the vertical integral (over the wave zone) of the Euler equations (in three dimensions) leads to an apparent force on the mean flow towards the line of maximum U of amount $\mathbf{M} \times \operatorname{curl} \mathbf{U} - \mathbf{U}\nabla . \mathbf{M}$, where $\mathbf{U}$ is the current ($\mathbf{U} = U\mathbf{i}$ in the applications) and $\mathbf{M}$ is the wave momentum per unit area $\mathbf{M} = E\boldsymbol{\kappa}/\omega'$ (equal to the vertical integral of $\boldsymbol{p}$ for irrotational waves in the WKB approximation); E is the wave energy density ($= \frac{1}{2}\rho g a^2$, where a is the local wave amplitude), $\boldsymbol{\kappa}$ is the wavenumber vector, and ω' is the intrinsic frequency ($= Ng\kappa$). This force exists even if it is evaluated for the incident wave alone, neglecting the refraction caused by the weak current anomaly $U(y)$. (We shall return to this point later.) Thus, given a wave field and a current with vertical vorticity, a convergence force is exerted on the water within the wave zone. Since the waves apparently amplify near the maximum of U, it is assumed that wave breaking may occur preferentially in this neighbourhood, leading to a momentum transfer from the waves to the Eulerian-mean current and essentially in the x direction.

The forces exerted in the wave zone are assumed to exert a stress on the water below, where viscosity is accounted for; the stress in the x-direction is parametrized in an *ad hoc* way. Convection cells develop in this subsurface water as an instability – the wave breaking provides a coupling between the x-velocity component and the components in the plane normal to the x axis that is essential for the instability to occur.

Garrett (private communication) has pointed out connexions between his theory and that of CL that are worth elaboration. Nearly monochromatic infinitesimal waves have a wave momentum per unit area in infinitely deep water

$$\mathbf{M} = \frac{E}{c}\boldsymbol{\kappa} = \tfrac{1}{2}\rho\omega' a^2 \boldsymbol{\kappa} = \int_{-\infty}^{0} \rho \bar{\mathbf{u}}^{s}\, dz,$$

where $\bar{\mathbf{u}}^{s}$ is the Stokes drift. In the case of interaction of the waves with weak currents Garrett's convergence force, per unit area, simplifies (as he points out) to

$$\mathbf{M} \times \operatorname{curl} \mathbf{U} = \left(\rho \int_{-\infty}^{0} \bar{\mathbf{u}}^{s}\, dz\right) \times \operatorname{curl} \mathbf{U}.$$

This expression is the vertical integral of the vortex force $\rho(\bar{\mathbf{u}}^{s} \times \operatorname{curl} \bar{\mathbf{u}})$ in the CL theories for the case of currents depending only upon y (as Garrett assumes). Furthermore, in the limit $\hat{\epsilon} \to 0$ the CL equations seem amenable to analysis by the method of matched asymptotic expansions. The linearized treatment of the equations below the wave zone yields the same problem as Garrett found for the motions in the plane perpendicular to the wind; in one plausible treatment, the effect of the vortex force on the subsurface motion below arises from a matching with the wave zone, and takes the form of the stress Garrett (1976) described. The subsurface motions do not, however, by themselves indicate an instability. For instability to occur via the CL theory, one must examine the details within the wave zone and, in particular, vertical shear in the current cannot be ignored. As mentioned earlier, for instability to arise, Garrett was forced to invoke instead an added stress due to wave breaking.

Moen (1978) disagrees with the radiation-stress arguments central to Garrett's theory; he nevertheless adopts Garrett's wave refraction solution as a central element in his theory and constructs a model with wave effects distributed throughout the watei depth. We refer to Moen's thesis for the specifics of his criticisms of Garrett's theory (although we do not find Moen's arguments against it to be persuasive in themselves). We note at this point that a logical inconsistency absent in Garrett's model now enters by the adoption of a wave field based upon ignoring the depth dependence of the current, since the Moen model purports to include depth dependence.

The weak vorticity acquired during refraction from the current $U(y)$ plays an essential role in Moen's construction. Using Garrett's solution, the increase in wave amplitude near maxima of the current $U(y)$ is noted. Moen then claims to prove that wave fields with spatial variations of wave kinetic energy necessarily carry vorticity, and to build a theory in which Langmuir circulations derive from variations of wave kinetic energy. This is demonstrably incorrect in principle. In actual use, however, Moen's application of Garrett's wave field essentially employs a means of calculating wave-induced vorticity fluctuations that differs from the CL theories; from that point, the method of approach is similar to that of CL and the final equations are identical to a simplified form of the CL equations.

4. Recapitulation and concluding remarks

We have shown that the CL equations are the proper approximation to the AM GLM equations for wave–mean flow interactions provided that the waves are of small amplitude and irrotational to lowest order in wave slope. No other approximations are required to effect the reduction AM $\rightarrow$ CL (although one must have a large wave Reynolds number and a rotational current weak compared to orbital speeds in the irrotational waves – these conditions, however, are necessary for the assumptions on the wave field to hold).

One is, of course, free to choose a small-amplitude irrotational wave field with characteristics quite different from the class of waves considered so far in the CL Langmuir circulation theories, but the effect of different choices is simply to alter the Stokes drift term appearing in the CL equations.

Some of the connexions and contrasts between the CL theory of Langmuir circulations, and the theories of Garrett (1976) and of Moen (1978), which introduce additional elements into wave–current interaction models of Langmuir circulations, have also been pointed out.

The CL theories do not require an explicit calculation of modifications of the surface-wave field. Wave refraction is a second-order effect under the CL hypotheses, which contemplate Langmuir circulation currents weak compared to wave particle speeds and horizontal spatial current variations on a scale comparable to the wavelengths of the dominant surface waves. The wave-refraction solution, on the other hand, upon which Garrett's and Moen's theories are based, is formally valid only for surface waves with wavelength *small* (the WKB approximation underlying kinematic wave theory) compared to Langmuir cell spacing. Observed windrows clearly exist when $(\kappa L)^{-1}$ is not small. Of course, the important question is not the formal validity of a calculation method requiring $(\kappa L)^{-1} \ll 1$, but the degree to which wave amplitude variations actually occur as a result of current patterns associated with the Langmuir circulation field. This question is currently under consideration by J. Smith at Dalhousie University using calculations not subject to the restrictions inherent in kinematic wave theories. (Partial wave reflexions – which are not represented in kinematic, or ray, theories – may lead to *reduced* amplitudes at the maximum of the current anomaly $U(y)$, instead of enhanced amplitudes.) Once the modification of the wave field caused by interaction with velocity fields characteristic of those in Langmuir circulations is better understood, it may be possible to assess the dynamical consequences of the second-order wave effects upon the mean flows.

In his analysis of field data from Lake George, Myer (1971) reported noticeable enhancement of wave amplitudes at lines of surface convergence in Langmuir circulations. This observation seems to be one motivation for Garrett's theory, which gives a theoretical explanation of it. It seems that a firm correlation between windrows and enhanced wave amplitudes remain to be established, however. Myer (1971) does not give sufficient information in his report to establish whether such a correlation is statistically significant for the data presented, whether enhanced wave activity may be dynamically necessary to create or maintain the circulations, or to allow one to conclude that such a correlation is invariably present in Langmuir circulations. We know, from the controlled experiments of Faller (1978) and others, that phenomena resembling Langmuir circulations can be created in the laboratory with a *minimum* in wave kinetic energy occurring along lines of surface convergence, in apparent agreement with the original CL mechanism explored in Craik & Leibovich (1976). Thus, a maximum wave kinetic energy coincident with surface convergence has been experimentally proved not to be necessary. In the second, or instability, CL mechanism, coherent variations of wave kinetic energy are not required for the initial formation of Langmuir circulations. The CL instability mechanism (which, according to the fully nonlinear computations in Leibovich & Paolucci, 1980, leads to LC patterns consistent with observations), on the other hand, does not preclude coherent variations in wave kinetic energy, provided that their dynamical effects on the mean current are of secondary importance.

Experiments under carefully controlled conditions and comparisons with computations carried out for competing theories using experimental conditions as input are feasible; they are needed to test the validity of available theoretical ideas, and to clarify the parametric range in which the theories provide useful representations of observed Langmuir circulation phenomena.

This paper has benefited from comments of A. D. D. Craik, C. J. R. Garrett, M. E. McIntyre and J. Moen. I am grateful to them all for their constructive criticism of the original draft of this paper.

This work was supported by the NSF Physical Oceanography Program under Grant OCE 77-04482.

REFERENCES

ANDREWS, D. G. & MCINTYRE, M. E. 1978 An exact theory of nonlinear waves on a Lagrangian-mean flow. *J. Fluid Mech.* **89**, 609–646.

BRETHERTON, F. P. 1971 The general linearized theory of wave propagation. *Lectures in Applied Mathematics*, vol. **13**, 61–102. American Mathematical Society.

CRAIK, A. D. D. 1970 A wave-interaction model for the generation of windrows. *J. Fluid Mech.* **41**, 801–821.

CRAIK, A. D. D. 1977 The generation of Langmuir circulations by an instability mechanism. *J. Fluid Mech.* **81**, 209–223.

CRAIK, A. D. D. & LEIBOVICH, S. 1976 A rational model for Langmuir circulations. *J. Fluid Mech.* **73**, 401–426.

FALLER, A. J. 1978 Experiments with controlled Langmuir circulations. *Science* **201**, 618–620.

GARRETT, C. J. R. 1976 Generation of Langmuir circulations by surface waves – a feedback mechanism. *J. Mar. Res.* **34**, 117–130.

HASSELMANN, K. 1970 Wave-driven inertial oscillations. *Geophys. Fluid Dyn.* **1**, 463–502.

HUANG, N. E. 1979 On the surface drift currents in the ocean. *J. Fluid Mech.* **91**, 191–208.

LEIBOVICH, S. 1977*a* On the evolution of the system of wind drift currents and Langmuir circulations in the ocean. Part 1. Theory and averaged current. *J. Fluid Mech.* **79**, 715–743.

LEIBOVICH, S. 1977*b* Convective instability of stably stratified water in the ocean. *J. Fluid Mech.* **82**, 561–585.

LEIBOVICH, S. & PAOLUCCI, S. 1980 The Langmuir circulation instability as a mixing mechanism in the upper ocean. *J. Phys. Oceanogr.* (to appear).

LEIBOVICH, S. & RADHAKRISHNAN, K. 1977 On the evolution of the system of wind drift currents and Langmuir circulations in the ocean. Part 2. Structure of the Langmuir vortices. *J. Fluid Mech.* **80**, 481–507.

LEIBOVICH, S. & ULRICH, D. 1972 A note on the growth of small scale Langmuir circulations. *J. Geophys. Res.* **77**, 1683–1688.

MOEN, J. 1978 A theoretical model for Langmuir circulations. Ph.D. thesis, University of Southampton.

MYER, G. E. 1971 Structure and mechanism of Langmuir circulations on a small inland lake. Ph.D. dissertation, State University of New York at Albany.

PHILLIPS, O. M. 1966 *The Dynamics of the Upper Ocean*. Cambridge University Press.