A multi-wavenumber theory for eddy diffusivities and its application to the southeast Pacific (DIMES) region

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ABSTRACT

A multi-wavenumber theory is formulated to predict eddy diffusivities. It expands on earlier single-wavenumber theories and includes the wide range of wavenumbers encompassed in eddy motions. In the limiting case in which ocean eddies are only composed of a single wavenumber, the multi-wavenumber theory is equivalent to the single wavenumber theory, and both show mixing suppression by the mean flow. The multi-wavenumber theory was tested in a region of the Southern Ocean (70°S – 45°S, 110°W – 20°W) that covers the Drake Passage and includes the tracer/float release locations during the Diapycnal and Isopycnal Mixing Experiment in the Southern Ocean (DIMES). Cross-stream eddy diffusivities and mixing lengths were estimated in this region from the single-wavenumber theory, from the multi-wavenumber theory, and from floats deployed in a global 1/10° Parallel Ocean Program (POP) simulation. Compared to the single-wavenumber theory, the horizontal structures of cross-stream mixing lengths from the multi-wavenumber theory agree better with the simulated float-based estimates at almost all depth levels. The multi-wavenumber theory better represents the vertical structure of cross-stream mixing lengths both inside and outside the Antarctica Circumpolar Current (ACC). Both the single-wavenumber and multi-wavenumber theories predict the horizontal structures of cross-stream diffusivities, which resemble the eddy kinetic energy patterns.

1. Introduction

Eddies are not explicitly resolved in standard-resolution centennial-scale global climate simulations; however, these simulations are sensitive to the representation of eddy mixing processes (e.g. Danabasoglu and Marshall 2007). This sensitivity has motivated efforts to estimate eddy mixing rates in the ocean that aim to reveal mixing processes leading to improved eddy parameterizations (e.g. Gille et al. 2012). Both float-based and tracer-based diagnostic approaches have been employed to estimate diffusivities in the tropical North Atlantic (e.g. Banyte et al. 2013), western boundary currents (e.g. Chen et al. 2014), the Southern Ocean (e.g. LaCasce et al. 2014; Tulloch et al. 2014), and the global surface (e.g. Abernathey and Marshall 2013).

Understanding the estimated diffusivity patterns is a necessary step toward improving eddy parameterization schemes. A common approach is to interpret the mixing length $L_{mix}$ instead of the diffusivity itself,

$$L_{mix} = \frac{\kappa}{\Gamma u_{rms}},$$

(1)

where $\Gamma$ denotes the $O(1)$ mixing efficiency, $u_{rms}$ the eddy velocity magnitude and $\kappa$ eddy diffusivity (e.g. Taylor 1915; Klocker and Abernathey 2014).

Bates et al. (2014) pointed out that ocean studies often assume that $L_{mix}$ is of the same order of magnitude as the eddy size (e.g. Holloway 1986; Haine and Marshall 1998). This assumption is reasonable in an ocean without mean flows (Klocker and Abernathey 2014). In regions of the real ocean that contain mean flows, such as the Antarctica

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Circumpolar Current (ACC) and western boundary currents, eddy mixing in the cross-mean flow (cross-stream) direction can be suppressed when eddies propagate relative to the mean flow. This concept dates back to Bretherton (1966) and Green (1970), and can be interpreted as follows. If eddies are stationary relative to the mean flow, eddies have sufficient time to stir and mix the same tracers, which also move with the mean flow; on the other hand, if the eddies propagate relative to the mean flow, eddies do not mix the same tracers, and mixing is suppressed (Klocker and Abernathey 2014).

Analytical formulas exist to represent the suppression of cross-stream mixing length by the mean flow (e.g. Green 1970; Killworth 1997; Ferrari and Nikurashin 2010; Klocker et al. 2012a). Ferrari and Nikurashin (2010) (the F-N theory) proposed that

$$L_{mix,\perp} = \frac{L}{1 + k_{edd}^2 \gamma^2 (C_w - |U|)^2},$$

where $L_{mix,\perp}$ is the cross-stream mixing length and $|U|$ denotes the mean flow magnitude. The terms $k_{edd}$, $L$, and $\gamma$ are the eddy wavenumber, eddy length scale, and the reciprocal of eddy decorrelation time, respectively. The eddy phase speed along the mean flow direction is denoted by $C_w$ and positive $C_w$ corresponds to upstream propagation (Chen et al. 2014). Surface eddy diffusivities from the F-N theory agree with those from altimetry (e.g. Ferrari and Nikurashin 2010; Klocker and Abernathey 2014; Bates et al. 2014). The F-N theory also predicts the three-dimensional structure of float-based eddy diffusivities in the intense Kuroshio Extension jet area (Chen et al. 2014). On the other hand, the theory appears to break down in regions where large topographic gradients occur (e.g. Griesel et al. 2010; Naveira-Garabato et al. 2011; Chen et al. 2014). Bates et al. (2014) found that the F-N theory also is not effective in predicting the observed vertical structures of eddy diffusivities from the US/UK field program Diapycnal and Isopycnal Mixing Experiment in the Southern Ocean (DIMES; e.g. Ledwell et al. 2011) or from the North Atlantic Tracer Release Experiment (NATRE; e.g. Ledwell et al. 1998).

The break down of the F-N theory in some ocean scenarios is unsurprising, considering that it is built on a number of assumptions that are violated in the ocean, including a flat bottom, spatially and temporally constant mean flow, and the scale separation between the mean flow and eddies (e.g. Ferrari and Nikurashin 2010). One assumption upon which these theories, including the F-N theory, are based is that eddies only contain a single wave corresponding to the most unstable mode, or that eddies are dominated by a single wave (e.g. Green 1970; Killworth 1997; Ferrari and Nikurashin 2010; Klocker et al. 2012a). In fact, the frequency-wavenumber spectra of sea surface height is a broad continuum with no robust peaks, indicating that the oceanic eddy field includes motions over a broad range of wavenumbers and frequencies (e.g. Wunsch 2010; Wortham 2013; Wortham and Wunsch 2014). Even a single Gaussian vortex can be decomposed into wave motions with a range of wavenumbers and frequencies (e.g. Chen 2013). In the mid-latitude ocean interior away from western boundary currents, the phase speeds for all the dominant wavenumbers are roughly the same; however, in the Gulf Stream, the Kuroshio Extension and the ACC, the dominant waves are dispersive, and both eastward and westward propagating waves are non-negligible [see Fig. 2-16 from Wortham (2013)].

Assuming that internal waves are small-amplitude Gaussian random processes, Holmes-Cerfon et al. (2011) developed a formula for one-particle horizontal diffusivity based on the internal wave spectra. Inspired by Holmes-Cerfon et al. (2011) and motivated by the discrepancy between the single-wavenumber assumption and the observed broad-banded oceanic spectra, here we formulate a multi-wavenumber theory (the M-W theory) for eddy diffusivities in the mean flow (section 2). The theory can be derived from both float-based and tracer-based diffusivity formulas (sections 2.a and 2.b), and it links diffusivities with the frequency-wavenumber spectra of the Eulerian eddy velocity fields.

We choose the DIMES region (70°S–45°S, 110°W – 20°W) as a testbed for the M-W theory (Fig. 1). It covers both the region upstream of the Drake Passage, where the floats and tracers were released in the DIMES experiment (e.g. Ledwell et al. 2011; Watson et al. 2013; Sheen et al. 2013; LaCasce et al. 2014; Tulloch et al. 2014), and the region downstream of the Drake Passage (Scotia Sea), where eddy kinetic energy (EKE) is larger and mixing is more intense than upstream of the Drake Passage. Eddy diffusivities are estimated at high spatial resolution from numerical floats deployed in a global eddy mixing model. These float-based diffusivities are used not only to test the relevance of the single-wavenumber and multi-wavenumber theories, but also to help put the sparse mixing observations in the DIMES region into a larger spatial context.

This paper is organized as follows: section 2 introduces the M-W theory about eddy diffusivities and illustrates its consistency with the F-N theory in the single-wavenumber limit. Section 3 describes the configuration of the global eddy mixing model with numerical floats, which we use to test the applicability of our theory. Section 4 presents eddy diffusivities in the DIMES area from the numerical floats in the eddying model. Section 5 provides the predicted diffusivities from both the F-N and M-W theories, and compares them with the float-based diffusivities. Sections 6 and 7 provide the discussion and summary, respectively.
2. The multi-wavenumber theory from both float and tracer perspectives

Here we derive a multi-wavenumber (M-W) theory for the diffusivity tensor and cross-stream diffusivities from both the float and tracer perspectives. Its consistency with the F-N theory in the single-wavenumber scenario is demonstrated in appendix A.

a. The multi-wavenumber theory derived from Lagrangian eddy diffusivities

1) DIFFUSIVITY TENSOR IN AUTOCORRELATION FORM

Lagrangian eddy diffusivities can be estimated from numerical floats. We start from the diagnostic formulas, involving the integral over the autocovariance function (Davis 1991; Griesel et al. 2010, 2014a; Chen et al. 2014):

\[
\kappa^L_{ij}(x) = \lim_{\tau \to \infty} \langle \kappa^L_{ij}(x, \tau) \rangle = \lim_{\tau \to \infty} \int_0^\tau d\tilde{\tau} \langle u'_{i,L}(t_0|x, t_0) u'_{j,L}(t_0 + \tilde{\tau}|x, t_0) \rangle,
\]

where \( \kappa^L_{ij}(x) \) is the value of the diffusivity tensor \( \kappa^L_{ij}(x, \tau) \) in the limit as time lag \( \tau \) goes to infinity and \( \langle \cdot \rangle \) denotes the ensemble average of many pseudo float trajectories with center positions passing \( x \). Here \( u'_{i,L}(t_0 + \tau|x, t_0) \) denotes the residual velocity in the \( i \) direction at time \( t_0 + \tau \) for the float that passes position \( x \) at time \( t_0 \). Residual velocities denote the float velocities subtracted from the local “time-mean” Eulerian velocity at the float position.

Following Klocker et al. (2012a), we assume that (a) the mean flow varies on spatial scales larger than eddies and thus the mean flow vector \( U \) is approximately a constant; and (b) the eddy velocity magnitude is much smaller than the mean flow magnitude. Therefore, to leading order floats move only with the mean flow. Consequently, the Lagrangian velocities \( u'_{i,L} \) can be obtained from the Eulerian velocities:

\[
u'_{i,L}(t_0|x, t_0) = u'_{i}(x, t_0), \quad \text{and} \quad u'_{i,L}(t_0 + \tilde{\tau}|x, t_0) = u'_{i}(x + U \tilde{\tau}, t_0 + \tilde{\tau}),
\]

where \( u'_{i} \) denotes the Eulerian eddy velocities in the \( i \) direction.

Following Klocker et al. (2012a), we set the starting time of the float \( t_0 \) to be zero, without loss of generality. Then substituting Eq. (4) into Eq. (3),

\[
\kappa^L_{ij}(x) = \lim_{\tau \to \infty} \int_0^\tau d\tilde{\tau} \langle u'_{i}(x + U \tilde{\tau}, t)|_{\tilde{\tau}=\tau} u'_{j}(x + U \tilde{\tau}, t)|_{\tilde{\tau}=\tau} \rangle.
\]

We define the Eulerian eddy velocities at position \( x + U t \) at time \( t \) as

\[
\mathcal{V}'(x, t) = u'_{i}(x + U t, t).
\]

then we obtain from Eq. (5),

\[
\kappa^L_{ij}(x) = \lim_{\tau \to \infty} \int_0^\tau d\tilde{\tau} \langle \mathcal{V}'(x, t)|_{\tilde{\tau}=\tau} \mathcal{V}'(x, t)|_{\tilde{\tau}=\tau} \rangle.
\]

Assuming that the eddy statistics are temporally stationary, we have

\[
\kappa^L_{ij}(x) = \frac{1}{2} \lim_{\tau \to \infty} \int_{-\tau}^\tau d\tilde{\tau} \langle \mathcal{V}'(x, t)|_{\tilde{\tau}=\tau} \mathcal{V}'(x, t)|_{\tilde{\tau}=\tau} \rangle.
\]

2) DIFFUSIVITY TENSOR IN SPECTRAL FORM

The derivation so far is similar to that of Klocker et al. (2012a), who provided a derivation for the F-N theory from the Lagrangian perspective. The second part of the derivation (in this subsection) diverges from Klocker et al. (2012a), in order to develop a multi-wavenumber mixing
formula. Using the one-dimensional cross-correlation theorem (e.g. Weisstein 2014), Eq. (8) leads to

$$\kappa^{L\omega}_{ij}(x) = \frac{1}{2} \lim_{\omega' \to 0} \langle \mathcal{W}_{ij}(\omega') \mathcal{W}_{ij}^*(\omega') \rangle$$

$$\approx \frac{1}{2} \cdot \left( \mathcal{W}_{i}(0) \mathcal{W}_{i}^*(0) \right) = \frac{1}{2} S_{y_{ij},y_{ij}}(\omega', x).$$

(9)

where $\hat{}$ is the Fourier transform, * denotes the complex conjugate, and $\omega'$ denotes frequency. Because the Fourier transform of a real variable is real at zero frequency, $\langle \mathcal{W}_{i}(0) \mathcal{W}_{i}^*(0) \rangle$ is real, even when $i$ and $j$ differ. Following Randel and Held (1991), the one-dimensional cross-spectrum of variables $\alpha$ and $\beta$ at position $x$, $S_{\alpha,\beta}(\omega', x)$, is defined as

$$S_{\alpha,\beta}(\omega', x) = \Re \langle \hat{\alpha}(\omega') \hat{\beta}^*(\omega') \rangle.$$

(10)

Following conventional notation (e.g. Ferrari and Wunsch 2010), we then define $S_{\alpha,\beta}(k', l', \omega', x)$ as the three-dimensional cross-spectrum in an oceanic patch centered at $x$

$$S_{\alpha,\beta}(k', l', \omega', x) = \Re \langle \hat{\alpha}(k', l', \omega', x) \hat{\beta}^*(k', l', \omega') \rangle.$$

(11)

Here we use a rotated coordinate system with wavenumber $l'$ perpendicular to the mean flow and wavenumber $k'$ aligned with and pointing in the mean flow direction.

To obtain enough ensembles (float trajectories), we often estimate diffusivities averaged over a selected oceanic patch (using either geographic or adaptive bins), rather than carrying out a point-wise estimate. The spatial average of the one-dimensional spectra over the oceanic patch can be obtained from the three-dimensional spectra,

$$\overline{S_{\alpha,\beta}(\omega', x)}_{\text{patch}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{\alpha,\beta}(k', l', \omega', x) dk'dl',$$

(12)

where $\overline{\cdot}_{\text{patch}}$ denotes the spatial average over the patch. Eqs. (9) and (12) lead to the diffusivity tensor averaged over the selected oceanic patch centered at $x$:

$$\overline{\kappa^{L\omega}_{ij}(x)}_{\text{patch}} = \frac{1}{2} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{y_{ij},y_{ij}}(k', l', \omega', x) |\omega'| dk'dl'.$$

(13)

In practice, $\overline{\kappa^{L\omega}_{ij}(x)}_{\text{patch}}$ is mostly easily diagnosed from the Eulerian eddy velocities $u'$, rather than from $\mathcal{W}'_{ij}$. Using the spectral analysis technique of Chen (2013), we obtain from Eq. (6),

$$\mathcal{W}_{i}(k', l', \omega') = \hat{u}_{i}(k', l', \omega' + |U|k'),$$

(14)

where $|U|$ is the magnitude of the mean flow vector. Therefore,

$$S_{y_{ij},y_{ij}}(k', l', \omega', x) = S_{u_{ij},u_{ij}}(k', l', \omega' + |U|k', x).$$

(15)

Substituting Eq. (15) into Eq. (13) leads to our multi-wavenumber formula for the diffusivity tensor averaged over the patch, involving the three-dimensional spectrum of Eulerian eddy velocities at the patch:

$$\overline{\kappa^{L\omega}_{ij}(x)}_{\text{patch}} = \frac{1}{2} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{u_{ij},u_{ij}}(k', l', |U|k', x) dk'dl'.$$

(16)

Using the above procedure, we can also obtain the diffusivity tensor averaged along a slice aligned with the mean flow direction and centered at $x$ (see Fig. 6c):

$$\overline{\kappa^{L\omega}_{ij}(x)}_{\text{slic}} = \frac{1}{2} \cdot \int_{-\infty}^{\infty} S_{u_{ij},u_{ij}}(k', |U|k', x) dk',$$

(17)

where $\overline{\cdot}_{\text{slic}}$ denotes the average along the slice, and $S_{u_{ij},u_{ij}}(k', \omega', x)$ is the two-dimensional cross-spectrum along the slice.

3) CROSS-STREAM DIFFUSIVITIES IN SPECTRAL FORM

The full diffusivity tensor has received much attention in eddy parameterizations and tracer transport studies (e.g. Bachman and Fox-Kemper 2013; Griffies 1998; Plumb and Mahlman 1987). However, previous investigations of the role of the mean flow in mixing have focused primarily on cross-stream mixing (e.g. Ferrari and Nikurashin 2010; Griesel et al. 2014a), which drives the eddy-induced meridional overturning circulation in the Southern Ocean (Tulloch et al. 2014). For consistency with recent work on cross-stream mixing, we next present cross-stream diffusivities in the spectral form. Subsequent sections focus on testing the validity of the M-W theory in predicting cross-stream diffusivities. For brevity, in this paper we will not discuss the full diffusivity tensor.

When the $i$ and $j$ components in Eq. (3) both represent the cross-stream direction, we obtain the float-based formula for cross-stream eddy diffusivities:

$$\overline{\kappa^{L\omega}_{ij}(x)}_{\text{cross}} = \lim_{\tau \to 0} \overline{\kappa_{ij}(x, \tau)} = \lim_{\tau \to 0} \int_{0}^{\tau} d\tau \langle u'_{ij}(t_0)|x, t_0) u'_{ij}(t_0 + \tau) |x, t_0) \rangle,$$

(18)

where $u'_{ij}(t_0 + \tau) |x, t_0)$ denotes the residual velocity in the cross-stream direction at time $t_0 + \tau$ for the float that passes position $x$ at time $t_0$.

Following sections 2.a(1) and 2.a(2), one can easily obtain cross-stream diffusivities in spectral form, again by using cross-stream components for both the $i$ and $j$ direction in Eqs. (16) and (17). The cross-stream diffusivity averaged over the patch centered at $x$ is

$$\overline{\kappa^{L\omega}_{ij}(x)}_{\text{cross}} = \frac{1}{2} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{u'_{ij},u'_{ij}}(k', l', |U|k', x) dk'dl',$$

(19)
where $S_{u',u'}(k',\omega',x)$ is the three-dimensional spectrum of cross-stream Eulerian eddy velocities over the patch. Similarly, the cross-stream diffusivity averaged over a slice aligned with the mean flow direction and centered at $x$ is

$$
\frac{\kappa_{k}^{\infty}(x)}{\kappa_{k}^{\text{slice}}}(x) = \frac{1}{2} \int_{-\infty}^{\infty} S_{u',u'}(k',|U|k',x)dk',
$$

where $S_{u',u'}(k',\omega',x)$ is the two-dimensional spectrum of cross-stream Eulerian eddy velocities along the slice. Since $\frac{\kappa_{k}^{\infty}(x)}{\kappa_{k}^{\text{slice}}}(x)$ has higher cross-stream resolution than $\kappa_{k}^{\infty}(x)$, we diagnose $\frac{\kappa_{k}^{\infty}(x)}{\kappa_{k}^{\text{slice}}}(x)$ in section 5. This will be denoted by $\kappa_{k}^{\text{multi}}(x)$ for short.

We find that the M-W theory is a natural extension of the single-wavenumber theory to a more realistic regime with multiple wavenumbers. As shown in appendix A, in the limit of a single wavenumber, cross-stream diffusivities from the M-W theory reduce to the diffusivities from the single-wavenumber F-N theory.

b. The multi-wavenumber theory derived from Eulerian eddy diffusivities

We can also obtain the M-W theory for diffusivities presented in section 2.a from an Eulerian diffusivity perspective. The tracer concentration $C$ satisfies

$$
\frac{\partial}{\partial t} C + \mathbf{u} \cdot \nabla C - \kappa_0 \nabla^2 C = 0,
$$

where $\mathbf{u}$ is the total velocity, and $\kappa_0$ is the molecular or numerical diffusivity of the tracer. The Eulerian eddy diffusivity tensor, $\kappa_{ij}^{\text{E}}$, is often defined as

$$
\overline{u'_i C'_j} = -\kappa_{ij}^{\text{E}} \frac{\partial}{\partial x_j} C,
$$

where the overbar denotes the ensemble average, the prime represents the deviation from the ensemble average, and $u'_i$ is the Eulerian velocity (e.g. Plumb and Mahlman 1987).

Consider the scenario from section 2.a: eddies are of small-amplitude compared to the mean flow and the system is spatially homogenous, with the spatial scale of the mean (e.g. mean flow and mean eddy flux) much larger than the eddy scale. Appendix B shows that, $\kappa_{ij}^{\text{E}}$ in this scenario is equivalent to the multi-wavenumber diffusivity tensor derived from the Lagrangian perspective [Eq. (16)], when $\kappa_0$ is very small.

3. Testing the theory: A global eddying model with numerical floats

We use a global eddying model to test the predictive skill of the M-W theory from section 2 in realistic contexts. The Parallel Ocean Program (POP) simulation used in this study is the same as that described by Chen et al. (2014). Here we briefly review the key model features. The domain is global and the grid has a nominal spatial resolution of $1/10^\circ$. In the vertical direction, the model is discretized into 42 vertical levels and the layer thickness decreases from 10 m at the surface to 250 m at the ocean bottom. The K-Profile Parameterization (Large et al. 1994) is used to represent vertical mixing in the upper ocean, and biharmonic diffusion parameterizes horizontal mixing at subgrid scales.

As summarized by Abernathey et al. (2013), a number of methods exist for estimating eddy diffusivities. Here we estimate Lagrangian eddy diffusivities from numerical floats (e.g. Davis 1987, 1991; Griesel et al. 2010). We deployed one million numerical floats at the beginning of the year 1994 uniformly over the entire globe at 23 vertical levels with a horizontal resolution of $0.25^\circ$ in latitude and $2.5^\circ$ in longitude. These floats were advected online by the three dimensional Eulerian velocity fields for a full year. The float properties, including position, velocity and density, were recorded at daily intervals.

Figure 1 shows our study domain, which extends from $70^\circ$S to $45^\circ$S and from $110^\circ$W to $20^\circ$W, roughly corresponding to the DIMES region. The barotropic streamlines, defined as $\psi_R = g f^{-1} \Phi$, are used as an indicator of the location of the time-mean ACC. Here $f$ is the Coriolis parameter, and $\Phi$ is the temporally averaged sea surface height over the year 1994-1995. The time-mean ACC is broad and roughly zonal west of Drake Passage. Once inside Drake Passage, it intensifies due to mass conservation; it then shifts northward to the east of Drake Passage.

**Fig. 2.** Eddy velocity magnitude ($\sqrt{u'^2 + v'^2}$) in m s$^{-1}$ at (a) 466 m and (b) 2625 m from the POP model. The barotropic streamlines are superimposed as black contours to indicate the location of the ACC. Here $u'$ and $v'$ are the deviation of total Eulerian velocity from the two-year (1994-1995) mean.
and becomes more spatially variable. Topographic variations are larger to the east of Drake Passage than they are to the west (Fig. 1). Figure 2 shows the eddy velocity magnitude at two selected depths. Though the eddy velocity magnitude gradually decreases with depth, its horizontal structures vary little with depth, with large magnitudes inside the ACC core.

The model represents eddies reasonably well in three ways. First, the small change in horizontal structure with depth, shown in Fig. 2, is consistent with an equivalent barotropic flow field in the Southern Ocean, as has been identified and employed in previous studies (e.g. Killworth and Hughes 2002; Firing et al. 2011; Klocker et al. 2012b). The horizontal structure of eddy velocity magnitude is consistent with that observed in altimetry (e.g. Farneti et al. 2010): eddy velocity magnitudes are large along the path of the ACC and are also larger to the east of Drake Passage than to the west. Second, in this area of the model, the domain average of the first baroclinic Rossby radius of deformation $R_d$ is 12 km, which is twice as large as the domain averaged grid size. Actually, $R_d$ is larger than the grid size in 88% of grid points in our study domain. Finally, as reviewed in section 1, in the spectra of sea surface height from altimetry, both westward and eastward propagating signals are significant in the Southern Ocean and Kuroshio Extension patches, where the eastward mean flow is intense (Wortham 2013). Similar features exist in the POP model (Fig. 3).

4. Eddy diffusivities in the DIMES region from numerical floats

a. Methodology

Chen et al. (2014) calculated critical layer depths and float-based eddy diffusivities in the Kuroshio Extension. This study uses a similar approach to estimate diffusivities from floats. As in Griesel et al. (2010) and Chen et al. (2014), we diagnose cross-stream diffusivities using Eq. (18). Residual velocities $u'_x$ represent the deviation of the float velocities from the local time-mean Eulerian velocities at the float positions. Here the time-mean refers to the temporal average over the year 1994-1995. We gather positions of floats within our study domain every other day. These serve as the center positions of the pseudotrajectories. For the analysis we consider float trajectories that extend 69 days backward in time and 69 days forward in time relative to the center position (Chen et al. 2014).

We use a clustering approach, described by Chen et al. (2014), to divide our domain into adaptive bins. These bins are irregularly distributed in space, and the number of pseudotrajectories in each bin is roughly the same (Fig. 4). This statistical uniformity leads to more converged diffusivity estimates than the geographic bin approach (Koszalka and LaCasce 2010; Chen et al. 2014). Using these bins, we estimated diffusivities at 11 depth intervals in the

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**Fig. 3.** Base 10 logarithm of the normalized frequency-zonal wavenumber spectrum of sea surface height in a patch extending over 30° longitude and 10° latitude, from (a) the Southern Ocean, (b) the Kuroshio Extension, and (c) the interior of the North Pacific. The center locations of the patches are (40°S, 15°W) for (a), (35°N, 160°E) for (b) and (30°N, 140°W) for (c). The spectra were obtained from the sea surface height from the POP model during 1997-1998, with the spatial and temporal mean over each patch removed. Black lines indicate the zero zonal wavenumber, and thus separate eastward and westward propagating signals in the spectra.

**Fig. 4.** (a) Colored dots indicate the location of centroids for each adaptive bin for diffusivity estimates at 400-600m. The color of the dots shows the number of pseudo tracks for each bin. (b) Histogram of the number of tracks per bin and red lines indicate the location of the number 400 and 600. The number of pseudo tracks for each bin is around 500 and the length of each pseudo track is 139 days.
upper 3000 m with layer thicknesses increasing from 45 m to 500 m with depth.

b. Numerical results

The horizontal patterns of float-based cross-stream diffusivities, as shown in Fig. 5, are similar to those of the eddy velocity magnitudes shown in Fig. 2, with large values inside the ACC. This indicates that the spatial patterns of eddy mixing are probably mostly controlled by EKE. Similar to the eddy velocity magnitude (Fig. 2), the horizontal structures of eddy diffusivities change little with depth, consistent with the equivalent barotropic nature of the flow field in the Southern Ocean. Diffusivities in the ACC east of the Drake Passage are larger than those to the west.

Figure 5 shows three-dimensional float-based estimates of diffusivities at high spatial resolution both west and east of Drake Passage. LaCasce et al. (2014) and Tulloch et al. (2014) estimated meridional isopycnal diffusivities, using tracer and float observations collected from the DIMES experiment. Given the quasi-zonal orientation of the ACC west of the Drake Passage, their meridional diffusivities are expected to be approximately equivalent to cross-stream diffusivities. They extrapolated the vertical structures of the meridional diffusivities from the DIMES observations using a regional eddying model. However, their estimates do not resolve horizontal structures.

Our cross-stream diffusivities are for the year 1994, which is roughly 15 years prior to the initial DIMES float and tracer deployment (e.g. Ledwell et al. 2011). We expect discrepancies between our results and the DIMES observations, due to the interannual variability of mixing, the sparse resolution of observations, numerical model and forcing errors, and the differences between cross-stream diffusivities and meridional diffusivities. Nonetheless, observations from Tulloch et al. (2014) and our modeling results agree both in the order of magnitudes of diffusivities and in the depth of the local peak. The spatially averaged cross-stream diffusivity in the ACC west of the Drake Passage (63°S – 55°S, 105°W – 75°W) in our model decreases from 590 ± 30 m² s⁻¹ at 500 m to 370 ± 10 m² s⁻¹ at 1150 m, and then increases to a local peak value (460 ± 10 m² s⁻¹) at 2150 m. Using DIMES observations and results from a regional model, Tulloch et al. (2014) estimated the meridional diffusivity west of the Drake Passage to be about 300-500 m² s⁻¹ in the upper kilometer, reaching a local peak value (900 m² s⁻¹) at 2000 m (Fig. 10 in Tulloch et al. 2014). Their local peak value is twice as large as our estimates. Note that Tulloch et al. (2014) estimated uncertainties only for the meridional diffusivity at 1500 m: 710 ± 260 m² s⁻¹. Error bars in our estimates are determined using 10 realizations of a bootstrapping technique by randomly sampling the grid-ded cross-stream diffusivities at selected depths. Reported uncertainties represent two times the standard error.

5. Eddy diffusivities in the DIMES region from multi-wavenumber theory: estimations and comparisons

This section examines diffusivities and mixing lengths from the M-W theory from section 2 and compares them with those from the F-N theory and from numerical floats.

a. Methodology

To assess whether we can obtain improved mixing estimates by extending the single-wavenumber theory to the multi-wavenumber scenario, we diagnose eddy diffusivities and mixing lengths from both the F-N and M-W theories. In contrast to Tulloch et al. (2014), we consider the entire DIMES domain, both upstream and downstream of Drake Passage.

We use \( \kappa_{\text{multi}} \) to denote the cross-stream diffusivities from the multi-wavenumber mixing formula Eq. (20) and \( L_{\text{mix, } \perp} \) for the corresponding mixing lengths. Similarly, \( \kappa_{\text{single}} \) and \( L_{\text{single, } \perp} \) respectively denote cross-stream diffusivities and mixing lengths from the F-N theory.

The diffusivity \( \kappa_{\text{multi}} \) depends on the frequency-wavenumber spectrum of cross-stream eddy diffusivities, \( S_{\text{u, } \perp} \). As illustrated in Fig. 6, we identify a slice centered at position \( \mathbf{x} \) with a length of 300 km, and then extract Eulerian eddy velocities along the slice during the year 1994 to form the Hovmöller diagram. \( S_{\text{u, } \perp} \) can be obtained from the extracted eddy velocities through the use of a two-dimensional Fourier transform. We then determine \( \kappa_{\text{multi}} \) by integrating \( S_{\text{u, } \perp} \) along \( \omega' = |\mathbf{U}| \) [Eq. (20)].

We diagnose \( L_{\text{mix, } \perp} \) from the F-N theory [Eq. (2)], using the same method as Chen et al. (2014). The dominant eddy wavenumber \( k_{\text{eddy}} \) is chosen to be the centroid of the eddy kinetic energy spectrum. The eddy size and the reciprocal of the eddy decorrelation time scale are determined from

\[
L(x, y, z) = \frac{2\pi}{k_{\text{eddy}}(x, y, z)}, \quad \text{and} \quad \gamma(x, y, z) = \frac{u_{\text{rms}}(x, y, z)}{2\Gamma L(x, y, z)}.
\]

As before, \( u_{\text{rms}} \) is the eddy velocity magnitude, and \( \Gamma \) denotes the mixing efficiency. We chose \( \Gamma \) to be 0.35, following Klocker and Abernathey (2014) and Chen et al. (2014). The phase speed \( C_{\text{p}} \) is determined from the Radon transform approach of Chen et al. (2014), and \( \mathbf{U} \) denotes the two-year (1994-1995) mean flow vector. We can then determine \( \kappa_{\text{single}} \) from the mixing length \( L_{\text{mix, } \perp} \) using Eq. (1):

\[
\kappa_{\text{single}}(x, y, z) = \Gamma u_{\text{rms}}(x, y, z)L_{\text{mix, } \perp}.
\]

Similarly, \( L_{\text{mix, } \perp} \) can also be obtained from \( \kappa_{\text{multi}} \) using Eq. (1).
Fig. 5. Cross-stream diffusivities at (a) 400-600 m and (b) 1900-2400 m on logarithmic scales. Dots indicate the location of the centroid of each adaptive bin for the diffusivity estimates. Black lines are those of the barotropic streamlines.

Fig. 6. (a1) the Hovmöller diagram of cross-stream velocity anomalies along the $x'$ direction extending from -150 km to 150 km, at 918 m in the year 1994. $x' = 0$ corresponds to (58°S, 74°W). As illustrated in (c), which is from Chen et al. (2014), $x'$ is oriented in the direction of mean flow at $x' = 0$, and velocity anomalies along the 300 km black line are extracted to form the Hovmöller diagram. Cross-stream velocity anomalies denote the deviation of velocities in the cross-mean flow direction from its mean over 1994-1995. (b1) is the same as (a1), but for a different location: 918 m at (62°S, 62°W). (a2) and (b2) are the frequency-wavenumber spectra of velocity anomalies shown in (a1) and (b1) respectively. Black lines in (a2) and (b2) denote $\omega = |U|k'$, where $|U|$ is the mean flow magnitude, $\omega$ denotes frequency, $k'$ is the wavenumber along the $x'$ direction, and positive $k'$ points in the mean flow direction.

b. Results

1) Comparison of Horizontal Structures

Figure 7 shows cross-stream diffusivities from the two theories and numerical floats. Consistent with float-based results, the M-W theory predicts large diffusivities occurring inside the ACC. The M-W theory also predicts large diffusivities along the eastern coast of the South Pacific, which is not the case for the float-based estimates. The F-N theory predicts horizontal patterns of diffusivities similar to the M-W theory, with large values inside of the ACC and along the eastern coast of the South Pacific.

Cross-stream mixing lengths have noisy horizontal patterns (Fig. 8). The M-W theory correctly predicts the large mixing lengths inside the ACC below 1000 m, but the F-N theory does not (right panel of Fig. 8). Both the M-W and F-N theories fail to capture effectively the large-scale structures of mixing lengths in the upper 1000 m (left panel of Fig. 8).

To quantify the predictive skill of the two theories, we correlated the float-based estimates with those predicted from the M-W and F-N theories (Fig. 9). Both theories capture the horizontal structures of cross-stream diffusivities: the correlation coefficients between the float-based estimates and the M-W theory predictions have roughly the same magnitude as those between floats and the F-N theory, with values of 0.4-0.6 (Fig. 9b). The similar skill of the two theories in predicting cross-stream diffusivity...
patterns is related to the fact that, in our study domain, the diffusivity patterns are mainly controlled by eddy kinetic energy. This indicates that obtaining accurate mixing lengths (e.g. Bates et al. 2014; Chen et al. 2014) may not...
Fig. 9. Correlation of (a) cross-stream mixing lengths and (b) diffusivities as a function of depth. In the legends, $L_{\text{mix},\perp}$ and $\kappa_\perp$ respectively denote mixing length and diffusivities in the cross-stream direction. Here $\beta_{\text{float, multi}}$ and $\kappa_{\text{float, single}}$ are respectively estimates in the cross-stream direction from numerical floats, M-W theory and F-N theory. $\kappa_{\text{float, single}}$ and $L_{\text{mix, single}}$ respectively denote eddy diffusivities and mixing lengths in the cross-stream direction. The two thick black contours of the barotropic streamlines in Fig. 10, with the value of $3 \times 10^4$ and $10^3$ m$^2$/s, approximately pass the northern and southern edge of the Drake Passage. They are chosen to be the boundaries of the ACC: the region inside the two black contours is defined to be inside the ACC; the rest of the area in our study domain is defined to be outside the ACC. Reported uncertainties here are two standard errors, obtained from 10 realizations of a bootstrapping technique by randomly sampling the grided values in the selected spatial domain.

Table 1. Percentage of the area that satisfies the criteria listed in the first column at 95% confidence level, in the entire domain, inside the ACC, and outside the ACC. Here we define $\text{corre}(\alpha, \beta)$ as the correlation between variables $\alpha$ and $\beta$ in the entire water column over a $3.2^\circ \times 3.2^\circ$ patch. Here $\beta_{\text{float, multi}}$ and $\kappa_{\text{float, single}}$ are respectively estimates in the cross-stream direction from numerical floats, M-W theory and F-N theory. $\kappa_{\text{float, single}}$ and $L_{\text{mix, single}}$ respectively denote eddy diffusivities and mixing lengths in the cross-stream direction. The two thick black contours of the barotropic streamlines in Fig. 10, with the value of $3 \times 10^4$ and $10^3$ m$^2$/s, approximately pass the northern and southern edge of the Drake Passage. They are chosen to be the boundaries of the ACC: the region inside the two black contours is defined to be inside the ACC; the rest of the area in our study domain is defined to be outside the ACC. Reported uncertainties here are two standard errors, obtained from 10 realizations of a bootstrapping technique by randomly sampling the grided values in the selected spatial domain.

<table>
<thead>
<tr>
<th>Region</th>
<th>entire domain (%)</th>
<th>inside ACC (%)</th>
<th>outside ACC (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{corre}(\kappa_{\text{float, multi}}, \kappa_{\text{float, single}}) &gt; 0$</td>
<td>91 ± 1</td>
<td>86 ± 1</td>
<td>95 ± 1</td>
</tr>
<tr>
<td>$\text{corre}(\kappa_{\text{float, multi}}, \kappa_{\text{float, single}}) &gt; 0$</td>
<td>55 ± 1</td>
<td>75 ± 1</td>
<td>41 ± 1</td>
</tr>
<tr>
<td>$\text{corre}(\kappa_{\text{float, single}}, \kappa_{\text{float, single}} &gt; 0$</td>
<td>77 ± 1</td>
<td>62 ± 2</td>
<td>87 ± 1</td>
</tr>
<tr>
<td>$\text{corre}(L_{\text{mix, multi}}, L_{\text{mix, single}}) &gt; 0$</td>
<td>73 ± 0</td>
<td>72 ± 1</td>
<td>74 ± 1</td>
</tr>
<tr>
<td>$\text{corre}(L_{\text{mix, single}}, L_{\text{mix, single}} &gt; 0$</td>
<td>34 ± 1</td>
<td>55 ± 1</td>
<td>19 ± 1</td>
</tr>
<tr>
<td>$\text{corre}(L_{\text{mix, single}}, L_{\text{mix, single}} &gt; 0$</td>
<td>77 ± 1</td>
<td>60 ± 1</td>
<td>88 ± 1</td>
</tr>
</tbody>
</table>

be essential to obtaining the correct diffusivity structures in the DIMES region.

When we examine the horizontal structures of cross-stream mixing lengths below 500 m, we see that the correlation coefficients between the M-W theory and floats are larger than those between the F-N theory and floats (Fig. 9a). The advantage of the M-W theory is more noticeable with increasing depth. Below 2000 m, the F-N prediction and float-based estimates decorrelate, whereas the correlation coefficient between the M-W theory predictions and float-based estimates increases to 0.3-0.4 (Fig. 9a).

2) Comparison of vertical structures

In order to assess how well the two theories predict the vertical structures of eddy mixing, we examined the correlation between float-based estimates and theoretical predictions throughout the water column (Fig. 10). In the regions where correlation coefficients are not significantly positive, the theory is considered not to have predictive skill. Larger positive correlation coefficients imply better predictive skill. Table 1 quantitatively compares the predictive skill of the two theories.

The F-N theory achieves better predictive skill inside the ACC than outside the ACC (Figs.10b1 and 10b2). It skillfully predicts vertical structures of cross-stream diffusivities in 75% of the area inside of the ACC, but in only 41% of the area outside of the ACC (Table 1). For vertical structures of cross-stream mixing lengths, the F-N theory provides skillful prediction in 55% of the area inside the ACC but only 19% of the area outside the ACC (Table 1).

The M-W theory better predicts the vertical structures of diffusivities and mixing lengths than the F-N theory both inside and outside the ACC (Fig. 10 and Table 1). In 91% of the study domain, the M-W theory predicts vertical
structures of diffusivities that are significantly correlated with float diffusivities. In contrast, the percentage is only 55% for the F-N theory. Table 1 reveals that, the predictive skill of the M-W theory outside the ACC is comparable to that inside the ACC. In contrast, the F-N theory is better inside the ACC than outside the ACC. Nonetheless, the M-W theory consistently outperforms the F-N theory. Overall, in roughly 90% of the area outside the ACC, the float-based estimates are better correlated with estimates from the M-W theory, than with estimates from the F-N theory (Table 1). In contrast, the percentage is only roughly 60% inside the ACC.

3) COMPARISON OF MAGNITUDES

Though the M-W and F-N theories predict the order of magnitude of eddy diffusivities and mixing lengths correctly, both theories overestimate the values (Fig. 11). In the upper 1000 m, the M-W theory overestimates the domain averaged diffusivities and mixing lengths more than the F-N theory; below 1000 m, the mismatch between the M-W theory and floats is smaller than it is between the F-N theory and floats.

6. Discussion

The main goal of the M-W theory was to modify the single-wavenumber theory to account for the multi-wavenumber feature of eddies. The single-wavenumber theory and the M-W theory are consistent in some respects. In the single-wavenumber limit, the M-W theory reduces to the single-wavenumber mixing formula of Ferrari and Nikurashin (2010) (appendix A). In addition, both theories are based on an assumption of spatial homogeneity, which means that horizontal variations in the mean flow and eddy properties are negligible. Finally, both theories explicitly illustrate the effect of the mean flow \( U \) on mixing.

In spite of their consistency, the single-wavenumber and M-W theories also have clear differences. The M-W theory more clearly depicts the ocean, in that oceanic eddies are composed of a range of wavenumbers, rather than a single dominant wavenumber. In addition, the M-W theory provides formulas for the diffusivity tensor and cross-stream diffusivities [Eqs. (17) and (20)]; in contrast the F-N theory only focuses on cross-stream diffusivities.

Since the M-W theory is in spectral form, it can be used to estimate the mixing rates induced by eddies at selected spatio-temporal scales. Separating the contributions of small-scale and large-scale eddies to mixing is useful, because in eddy-permitting models, large eddies are explicitly resolved, and only small eddies need to be parameterized (e.g. Fox-Kemper and Menemenlis 2008). Separating the contributions of low-frequency and high-frequency eddies to mixing is also useful, because in contrast with high-
frequency eddies, low-frequency eddies are dominated by banded structures (striations) leading to anisotropic mixing (e.g. Maximenko et al. 2005). The M-W theory [Eq. (20)] tells us that the contribution to cross-stream mixing from eddies with frequencies smaller than $\Omega_S$ is

$$\kappa_{\perp}^{\text{multi}}(x) = \frac{1}{2} \int_{-\Omega_S/|U|}^{\Omega_S/|U|} S_{u_u', u_u'}(k', |U|k', x) dk'. \quad (25)$$

The advantage of Eq. (25) is that it is computationally more efficient than some previous related frameworks (e.g. Chen 2013).

Our work on the M-W theory has several implications. First, the fact that the M-W theory outperforms the F-N theory in mixing length predictions indicates that it is useful to consider the multi-wavenumber regime in future development of mixing theories and eddy parameterizations. In addition, much effort has been devoted to characterizing and interpreting oceanic spectra in order to reveal the underlying processes of oceanic turbulence (e.g. Xu and Fu 2011). The M-W theory links velocity spectra with mixing, underscoring the importance of spectra for mixing. Finally, critical layer theory suggests that elevated values of mixing lengths occur at the critical layer depth, where the wave phase speed matches the mean flow magnitude (e.g. Bretherton 1966; Green 1970). Critical layer depth has been estimated either by identifying the phase speed of the fastest growing mode from linear stability analysis (e.g. Smith and Marshall 2009) or by identifying the dominant phase speed of the eddy field through the Radon transform (Chen et al. 2014). Each wavenumber in the eddy field corresponds to a different critical layer depth, and this work indicates that it is important to consider the contribution of all the waves in the eddy field to mixing. Therefore, future studies would potentially benefit from estimating a critical layer depth specific to each wavenumber.

Though successful in many respects, as shown in Fig. 11, the M-W theory like the F-N theory overestimates the domain averaged cross-stream mixing lengths by roughly 5-10 km. Since both theories overestimate the mixing length (Fig. 11), the single-wavenumber assumption probably is not the leading order explanation for the overestimation. When diagnosing the float-based diffusivities, in order to reduce the dispersion caused by the mean flow shear, as in Griesel et al. (2010), we calculate residual velocities from Eq. (18) by subtracting the local mean flow rather than the spatially uniform mean flow. However, both F-N and M-W theories assume the mean flow to be constant and thus do not include a mechanism to reduce the dispersion due to mean flow shear. Therefore, the spatial homogeneity assumption inherent in these theories might contribute to the overestimation, though detailed examination of this hypothesis remains to be done. Further improvements to the mixing theory probably should take into account the spatial inhomogeneity (e.g. horizontal shear in the mean flow), which can be induced by topographic effects, the localized formation of coherent vortices, etc.

It is not trivial to take into account the spatial inhomogeneity in eddy parameterizations. Appendix B illustrates analytically that, in an inhomogeneous system, where the mean tracer gradient and eddy fluxes vary over short spatial scales, eddy mixing depends on both the local and non-local mean tracer gradients. However, the concept of the eddy diffusivity itself, on which the F-N and M-W theories are built, is based on the assumption that eddy mixing and transport processes can be parameterized using the local mean tracer gradient.

While the shortcomings of the M-W theory identified in this study indicate that non-local eddy parameterization
schemes may ultimately prove valuable, the M-W theory nonetheless merits further assessment, particularly since the assumption of spatial homogeneity is reasonable in regions such as the mid-latitude ocean interior. In addition to cross-stream diffusivities, the M-W theory also provides predictions for the diffusivity tensor. The relevance of the M-W theory to mixing in other regions or other years has not yet been explored. Griesel et al. (2014b) diagnosed the diffusivities from the single-wavenumber theory using eddy parameters obtained from linear instability analysis, and then compared with those from numerical floats in the entire Southern Ocean. Similar to what we found in the DIMES region, they found that almost nowhere in the Southern Ocean did the single-wavenumber theory effectively predict the vertical structures of mixing. This suggests that it would be useful to assess whether the M-W theory provides better predictions of the vertical structure of mixing outside of the DIMES region.

7. Summary

Though oceanic eddies contain motions spanning a wide range of wavenumbers (e.g. Wunsch 2010), previous theories of eddy mixing have often been based on the assumption that eddies are composed of a single or dominant wave (e.g. Green 1970; Killworth 1997; Ferrari and Nikurashin 2010; Klocker et al. 2012a). Motivated by this discrepancy, we formulated a M-W theory of mixing, starting from the problem described by Klocker et al. (2012a). Our M-W theory, which can be derived from both Lagrangian and Eulerian perspectives, is based only on the mean flow and eddy velocity spectra, which are unambiguous and straightforward to calculate.

We chose the DIMES region to compare mixing theories. Eddy diffusivities and mixing lengths in the cross-stream direction were estimated using numerical floats deployed in a global eddying model. Cross-stream diffusivities are large inside the ACC, where eddy amplitudes are large. Horizontal structures of cross-stream diffusivities and eddy kinetic energy vary little with depth, indicating the relevance of the equivalent barotropic nature in this region. These float-based mixing estimates at high spatial resolution can serve as a context for observational results from the DIMES experiment.

The float-based mixing estimates were then compared with those from both the F-N and M-W theories. We found that the F-N and M-W theories have similar skill in capturing the horizontal structures of cross-stream eddy diffusivities, which are mainly controlled by eddy velocity magnitudes. Correlation analysis indicates that, compared to the F-N theory, the M-W theory is better at predicting both the horizontal and vertical structures of cross-stream mixing lengths. Therefore, a so-called dominant wave is insufficient to capture the mixing length properties in the DIMES area and possibly in other ocean regions as well. It is probably useful to continue considering the full range of waves in the ocean when developing new mixing theories or parameterization schemes.

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APPENDIX A

Consistency between the single- and multi-wavenumber theories in the single-wavenumber scenario

Here we illustrate that, in the single-wavenumber scenario, the cross-stream diffusivity from the multi-wavenumber theory (M-W theory) in section 2 is consistent with the single-wavenumber formula for cross-stream diffusivities from Ferrari and Nikurashin (2010) (F-N theory).

a. Review of the F-N theory

Following Flierl and McGillicuddy (2002), Ferrari and Nikurashin (2010) assumed that eddies are forced by stochastic time-varying forcing with a single wavenumber. They employed a surface quasi-geostrophic model, and defined the mean velocity $U$ and buoyancy $B$ as

$$U(z) = U_0 \frac{z + H}{H}, \quad B(y, z) = -\Gamma y + N^2 z, \quad (A1)$$

where $N$ is the buoyancy frequency, $\Gamma = fU_0/H$ denoting the horizontal buoyancy gradient, and $f$ is the Coriolis parameter. The eddy field satisfies

$$\frac{\partial}{\partial t} b + U_0 \frac{\partial}{\partial x} b - \Gamma \frac{\partial}{\partial x} \psi = \|\nabla\| Re \left[ r(t) e^{(kx+ly)} \right] - \gamma b, \quad (A2)$$

and

$$\partial_t^2 \psi + \frac{\partial^2}{N^2} \frac{\partial^2}{N^2} \psi + \frac{\partial^2}{N^2} \psi = 0, \quad z < 0, \quad (A3)$$

where $\psi$ is eddy streamfunction, $b$ is surface eddy buoyancy, $k$ zonal wavenumber, $l$ meridional wavenumber, $\gamma$ linear damping rate, and $r(t)$ white noise forcing with zero mean and autocorrelation function $\langle r(t) r(t') \rangle = \delta(t-t')$. $\|\nabla\|$ sets the forcing and eddy amplitudes.
Solutions satisfying Eqs. (A2) and (A3) can be written as
\[
\psi(x,y,z,t) = \frac{1}{\kappa} \Re \left\{ a(t) \exp[i(kx + ly) + \frac{N\kappa}{f} z] \right\}, \text{ and}
\]
\[
b(x,y,z,t) = N\kappa \psi(x,y,z,t),
\]
where \(\kappa\) is the wavenumber magnitude \(\sqrt{k^2 + l^2}\). Substituting Eq. (A4) into Eq. (A2) gives
\[
\frac{da}{dt} + \left[ \gamma + ik(U_0 - \frac{\Gamma}{N\kappa}) \right] a = \sqrt{\gamma} r(t), \tag{A5}
\]
The variable \(a\) can be obtained from Eq. (A5); then the eddy variables \(\psi\) and \(b\) are solved.

To solve the cross-stream diffusivity, they consider a passive tracer equation satisfying
\[
\frac{\partial}{\partial t} C' + U(z) \frac{\partial}{\partial x} C' + J(\psi, C') + \Gamma_c \frac{\partial}{\partial x} \psi = 0, \tag{A6}
\]
where \(C'\) is the tracer concentration and \(\Gamma_c\) the mean tracer gradient. They obtain the solution to \(C'\) from Eq (A6) by assuming \(C'\) has the same spatial structure as \(\psi\). Then the cross-stream Eulerian diffusivity can be obtained from
\[
\sqrt{\gamma} C' = \frac{\partial \psi}{\partial x} C' = -\kappa^E \frac{\partial}{\partial y} C. \tag{A7}
\]
Their final solution to \(\kappa^E\) in their single-wavenumber mixing model is
\[
\kappa^E_{\text{multi}} = \kappa^E = \frac{k^2}{\gamma^2 + k^2[C_w - U(z)]^2} EKE, \tag{A8}
\]
where \(C_w\) denotes the wave phase speed
\[
C_w = \left(1 - \frac{f}{NH\kappa}\right) U_0 = U_0 - \frac{\Gamma}{N\kappa}, \tag{A9}
\]
and \(EKE\) is the domain averaged eddy kinetic energy
\[
EKE = \frac{1}{2} \langle u'^2 + v'^2 \rangle = \frac{1}{2} \langle |\nabla \psi|^2 \rangle = \frac{1}{4} \frac{EKE}{\gamma^2} \exp\left(\frac{2N\kappa}{f} \right). \tag{A10}
\]

b. Cross-stream diffusivity from the multi-wavenumber theory in the single-wavenumber limit

As shown in Eq. (19), to obtain cross-stream diffusivities from the M-W theory, the key is to derive the frequency-wavenumber spectra of cross-stream eddy velocities. In the single-wavenumber scenario from Ferrari and Nikurashin (2010), we can obtain this spectrum from Eqs. (A4) and (A5).

Taking the Fourier transform of Eq. (A4) and multiplying by \(k^2\), we obtain the spectrum for cross-stream (in this case, meridional) velocities:
\[
S_{\psi}(k', l', \omega) = k^2 S_{\psi}(k', l', \omega') = \frac{1}{4} \int k^2 \exp \left(2N\kappa \frac{\omega}{f} \right) \left| \hat{a}(\omega') \right|^2 \frac{\delta(k - k', l - l')}{2}, \tag{A11}
\]
where \(\hat{a}\) is the Fourier transform, \(S_{\psi}\) denotes the spectra for variable \(\psi\), and \(\left| \hat{a}(\omega') \right|^2\) can be obtained through Fourier transform of Eq. (A5):
\[
\left| \hat{a}(\omega') \right|^2 = \frac{\gamma \left| \hat{\Gamma}(\omega') \right|^2}{\gamma^2 + \left| \omega' - k(U_0 - \frac{\Gamma}{N\kappa}) \right|^2}. \tag{A12}
\]

Substituting Eqs. (A11) and (A12) into Eq. (19) gives the cross-stream diffusivities from the M-W theory,
\[
\kappa^E_{\text{multi}} = \frac{1}{4} \int k^2 \exp \left(2N\kappa \frac{\omega}{f} \right) \frac{\gamma \left| \hat{\Gamma}(U_k) \right|^2}{\gamma^2 + \left| U_k - k(U_0 - \frac{\Gamma}{N\kappa}) \right|^2}. \tag{A13}
\]

Noting from Eq. (A9) that
\[
-(C_w - U) k = \left[U_k - k \left(U_0 - \frac{\Gamma}{N\kappa}\right) \right], \tag{A14}
\]
Using Eqs. (A14) and (A10), Eq. (A13) can be rewritten as
\[
\kappa^E_{\text{multi}} = \frac{1}{4} \frac{k^2}{\gamma^2 + k^2[C_w - U(z)]^2} \left[4EKE \cdot \left| \hat{\Gamma}(U_k) \right|^2 \right], \tag{A15}
\]
Recall that \(r(t)\) is the white noise forcing with zero mean and autocorrelation \(\langle r(t)r^*(t') \rangle = \delta(t - t')\); thus \(\left| \hat{\Gamma}(U_k) \right|^2 = 1 \) and \(\kappa^E_{\text{multi}} = \kappa^E_{\text{single}}\). Therefore, in the single-wavenumber limit, the cross-stream diffusivity formula from the M-W theory reduces to the single-wavenumber formula from Ferrari and Nikurashin (2010).

APPENDIX B

Derivation of the multi-wavenumber theory from tracers and flux-gradient relation

Here we show that the multi-wavenumber (M-W) theory from section 2.a can also be derived from an Eulerian diffusivity perspective using tracers. The mathematical symbols used below follow the convention of Eulerian diffusivities and only apply in this appendix.
We take the mean and eddy equations for the passive tracer $C$:

$$\frac{\partial}{\partial t} \overline{C} + \overline{u} \cdot \nabla \overline{C} - k_0 \nabla^2 \overline{C} = -\nabla \cdot \overline{F}$$  \(\text{(B1)}\)

and

$$\frac{\partial}{\partial t} C + (\overline{u} + u') \cdot \nabla C - k_0 \nabla^2 C = -u' \cdot \nabla \overline{C} + \nabla \cdot \overline{F},$$  \(\text{(B2)}\)

where $u$ denotes velocity, $k_0$ the molecular or numerical diffusivity of the tracer, and $F$ the eddy flux $u'C'$. Here $\tau$ denotes the ensemble average. For example, $\overline{u}$ and $u'$ denote mean and eddy velocities. We shall use ensemble averages with a suitably large number of ensembles.

If we define the Green’s function for each ensemble,

$$\left[ \frac{\partial}{\partial t} + (\overline{u} + u') \cdot \nabla - k_0 \nabla^2 \right] G'(x,t|x',t') = \delta(x-x') \delta(t-t'),$$  \(\text{(B3)}\)

we obtain

$$C'(x,t) = -\int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' G(x,t|x',t') u'_j(x',t') \frac{\partial \mathcal{C}(x',t')}{\partial x'_j}$$

$$+ \int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{J}_j(x,t|x',t') \frac{\partial \mathcal{F}_j}{\partial x_j},$$  \(\text{(B4)}\)

where $\int_{\mathcal{D}} dx'$ denotes integrating over the entire available spatial domain $\mathcal{D}$. From this, we find the ensemble averaged eddy flux

$$\mathcal{F}_i = \overline{u_i} \overline{C} = -\int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{J}_j(x,t|x',t') \frac{\partial \mathcal{C}(x',t')}{\partial x'_j}$$

$$+ \int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{H}_j(x,t|x',t') \frac{\partial \mathcal{F}_j}{\partial x_j},$$  \(\text{(B5)}\)

where

$$\mathcal{H}_j(x,t|x',t') = \overline{u'_i(x,t)} G(x,t|x',t') u'_j(x',t').$$  \(\text{(B6)}\)

This becomes an integral/differential equation for $\mathcal{F}$:

$$\mathcal{F}_i - \int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{J}_j(x,t|x',t') \frac{\partial \mathcal{F}_j}{\partial x'_j}$$

$$= -\int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{H}_j(x,t|x',t') \frac{\partial \mathcal{C}(x',t')}{\partial x'_j}.$$  \(\text{(B8)}\)

Note that the widely-used standard form of eddy parameterization is

$$\mathcal{F}_i = -\kappa_{ij}^E \frac{\partial \mathcal{C}(x',t')}{\partial x'_j},$$  \(\text{(B9)}\)

where $\kappa_{ij}^E$ is the Eulerian diffusivity tensor. Assuming spatial homogeneity, both $\mathcal{F}_j$ and $\frac{\partial \mathcal{C}(x',t')}{\partial x'_j}$ vary slowly spatially; thus the second term on the left hand side of the accurate form [Eq. (B8)] is roughly zero and $\frac{\partial \mathcal{C}(x',t')}{\partial x'_j}$ on the right hand side of the accurate form can be pulled out of the integral. Thus, the accurate form [Eq. (B8)] reduces to the standard form [Eq. (B9)] in the spatially homogeneous case.

In the spatially inhomogeneous case, the accurate form [Eq. (B8)] differs from the standard form [Eq. (B9)]. First, $\mathcal{F}_j$ from Eq. (B8) depends on both local and nonlocal mean tracer gradients. Second, we need to solve an integral equation for the flux. Though complicated, in principle, for many realizations, we should be able to solve the $G(x,t|x',t')$ equation [Eq. (B3)] numerically and average to find $\mathcal{H}_j(x,t|x',t')$ and $\mathcal{J}_j(x,t|x',t')$.

Now we consider an idealized case. As in section 2.a, we assume that (1) the mean flow velocity is much larger than the eddy velocity, that is, $\overline{u} \gg u'$; (2) the spatial scales of the mean flow and eddy flux are much larger than the eddies; in other words, the system is approximately spatially homogeneous. Our assumption of homogeneity implies that $\overline{u}$ is constant. We also assume that $k_0$ in Eq. (B1) is small. Under the small eddy amplitude and small $k_0$ assumptions, Eq. (B3) is reduced to

$$\left( \frac{\partial}{\partial t} + \overline{u} \cdot \nabla \right) G'(x,t|x',t') = \delta(x-x') \delta(t-t'),$$  \(\text{(B10)}\)

where $\overline{u}$ is the mean velocity. The eddy flux formula [Eq. (B5)] is reduced to

$$\mathcal{F}_i = \overline{u_i} \overline{C} = -\frac{\partial \mathcal{C}}{\partial x_j} \int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{J}_j(x,t|x',t').$$  \(\text{(B12)}\)

Therefore, in this idealized case, the Eulerian eddy diffusivity, defined by the widely-used formula [Eq. (B9)], is

$$\kappa_{ij}^E = \int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{J}_j(x,t|x',t').$$  \(\text{(B13)}\)

Substituting Eqs. (B6) and (B11) into Eq. (B13) leads to

$$\kappa_{ij}^E = \int_{-\infty}^{t} \int_{-\infty}^{t'} dx' dt' \mathcal{H}_j(x,t|x',t') \overline{u_i(t)} \overline{u_j(t)} dt'.$$  \(\text{(B14)}\)

Following section 2.a(1), we set $t$ to be zero without the loss of generality. Then we obtain from Eq. (B14),

$$\kappa_{ij}^E = \int_{-\infty}^{0} d\tau \mathcal{H}_j(x,t)|_{\tau=0} \cdot \mathcal{H}_j(x,t)|_{\tau=\tau},$$  \(\text{(B15)}\)
where $\mathcal{W}'(x,t)$ is defined in Eq. (6). Assuming that the eddy statistics are temporally stationary, we have

$$\kappa_{ij}^E = \frac{1}{2} \int_{-\infty}^{\infty} d\xi \mathcal{W}'(x,t)|_{\xi=0} \cdot \mathcal{W}'(x,t)|_{\xi=\xi}. \quad (B16)$$

Note that here denotes the ensemble average which is essentially the same as $\langle \cdot \rangle$ in the main text. Therefore, $\kappa_{ij}^E$ from Eq. (B16) is the same as $\kappa_{ij}^{L_{\infty}}$ from Eq. (8). Following the derivation from sections 2.a(2) and 2.a(3), we will obtain the multi-wavenumber formulas for the diffusivity tensor [Eq. (17)] and cross-stream diffusivities [Eq. (20)].

References


