Third recitation on Thursday April 16th

Reading assignment

Read lecture 4 Why integrals? and section 3.2 The Landau Symbols. I won’t have time to cover this material in class.

Problem 3.6 Asymptotic equivalence

True or false as \( x \to \infty \)

\[
\begin{align*}
(i) \; x + \frac{1}{x} & \approx x, \quad (ii) \; x + \sqrt{x} & \approx x, \quad (iii) \; \exp\left(x + \frac{1}{x}\right) & \approx \exp(x), \\
(iv) \; \exp\left(x + \sqrt{x}\right) & \approx \exp(x), \quad (v) \; \cos\left(x + \frac{1}{x}\right) & \approx \cos x, \quad (v) \; \frac{1}{x} & \approx 0?
\end{align*}
\]

Problem 3.8 — the definition of asymptoticity

Show from the definition of asymptoticity that

\[ e^{-1/x} \sim 0 + 0 x + 0 x^2 + 0 x^3 + \cdots \quad \text{as} \; x \downarrow 0. \]  

Note that \( e^{-1/x} \) is not \( \approx 0 \). Thus one really should insist on the distinction between \( \approx \) and \( \sim \). But most people don’t and just write \( \sim \) in both cases.

Problem 4.7 — Euler’s constant

Euler’s constant is defined by

\[ \gamma_E \overset{\text{def}}{=} -\Gamma'(1). \]  

(i) Show by direct differentiation of the definition of the \( \Gamma \)-function that:

\[ \gamma_E = - \int_0^\infty e^{-t} \ln t \, dt. \]  

(ii) Judiciously applying IP to the RHS, deduce that

\[ \gamma_E = \int_0^1 \frac{1 - e^{-t} - e^{-t^{-1}}}{t} \, dt. \]  

Problem 5.2 — IP medley

Using integration by parts to find \( x \to \infty \) asymptotic approximations of the integrals

\[ A(x) = \int_0^x e^{-t^4} \, dt, \quad B(x) = \int_0^x e^{+t^4} \, dt, \quad D(x) = \int_1^2 \frac{\cos xt}{t} \, dt. \]  

In each case obtain a two-term asymptotic approximation and exhibit the remainder as an integral. Explain why the remainder is smaller than the second term as \( x \to \infty \).
Figure 1: Upper panel is the exact integrand in (9) (the solid curve) and the Gaussian approximation (dashed). Lower panel compares the $F(x)$ obtained by numerical quadrature (solid) with the asymptotic approximation. The comparison is not great — the problem below asks you to calculate the next term in the asymptotic expansion and add that to the figure.

Second hand-in: due in class on Thursday April 24th

Problem 5.7 — IP and Watson’s lemma

Make sure you have read section 4.6. Complexification. There we evaluate $\text{Ai}(0)$ and encounter a special case, namely $n = 3$, of the integral

$$Z(n, x) \overset{\text{def}}{=} \int_0^{\pi/(2n)} e^{-x \sin n \theta} \, d\theta.$$  

(8)

In section 4.6 we use Jordan’s lemma to show that $\lim_{x \to \infty} Z(n, x) = 0$. This is not an asymptotic result because Jordan’s lemma does not precisely identify the rate at which $Z$ vanishes.

Convert $Z(n, x)$ to a Laplace transform and use Watson’s lemma to obtain the first three terms of the $x \to \infty$ asymptotic expansion.

Problem 6.12 — Laplace’s method (the version in the notes is garbled)

Find three terms in the $x \to \infty$ asymptotic expansion of

$$F(x) \overset{\text{def}}{=} \int_0^1 \exp \left( -\frac{xt^2}{1+t} \right) \, dt.$$  

(9)

Improve figure 1 by adding some higher-order approximations to the lower panel.
%% Laplace's method -- some steps have been replaced by ??????

```matlab
T = linspace(0,1); phi = T.^2./(1+T);
for x = [1 5 25 125]
    subplot(2,1,1)
    plot(T,exp(-x*phi))
    hold on
    plot(T,exp(-x*T.^2),'--')
end

xlabel('$t$', 'interpreter','latex','fontsize',16)
ylabel('Integrand', 'interpreter','latex')
text(0.8,0.77,'$x=1$', 'interpreter','latex','fontsize',12,'rotation',-9)
text(0.5,0.52,'$x=5$', 'interpreter','latex','fontsize',12,'rotation',-23)
text(0.28,0.35,'$x=25$', 'interpreter','latex','fontsize',12,'rotation',-35)
text(0.1,0.55,'$x=125$', 'interpreter','latex','fontsize',12,'rotation',-70)

x = linspace(0,50);
ExactIntegral = zeros(1,length(x));
nloop = 0;
for n=1:length(x)
    nloop = nloop + 1;
    F=@(v)exp(-x(nloop)*v.^2./(1+v));
    ExactIntegral(nloop) = quad(F,0,1);
end
subplot(2,1,2)
plot(x,ExactIntegral)
hold on
%sx = linspace(5,max(x)); % didn’t use this variable
plot(x,?????????,'--')
xlabel('$x$', 'interpreter','latex','fontsize',16)
ylabel('$F(x)$', 'interpreter','latex')
h = legend('quad','?????????')
set(h,'interpreter','latex')
axis([0 max(x) 0 1.05])
```