Sixth recitation on Thursday May 14th

Problem 13.6 — an elliptic integral

Find two terms in the expansion the elliptic integral

\[ K(m) \overset{\text{def}}{=} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}, \]  

as \( m \uparrow 1 \).

Problem 14.1 — a basic boundary layer problem

Find the leading-order uniformly valid boundary-layer solution to the Stommel problem

\[-(e^x h)_x = \epsilon h_{xx} + 1, \quad \text{with BCs} \quad f(0) = f(1) = 0.\]  

Do the same for

\[(e^x f)_x = \epsilon f_{xx} + 1, \quad \text{with BCs} \quad f(0) = f(1) = 0.\]  

Problem 14.4 — variable speed Stommel problem

Analyze the variable-speed Stommel problem

\[ \epsilon h'' + (x^a h)_x = -1, \quad \text{with BCs} \quad h(0) = h(1) = 0, \]  

using boundary layer theory. (The case \( a = 1/2 \) was discussed in the lecture.) How thick is the boundary layer at \( x = 0 \), and how large is the solution in the boundary layer? Check your reasoning by constructing the leading-order uniformly valid solution when \( a = -1, a = 1 \) and \( a = 2 \).

Problem 14.8 — a Burgers boundary layer

Find a leading-order boundary layer solution to the forced Burgers equation

\[ \epsilon h_{xx} + \left( \frac{1}{2} h^2 \right)_x = -1, \quad h(0) = h(1) = 0. \]  

Use \texttt{bvp4c} to solve this problem numerically, and compare your leading order solution to the numerical solution: see the figure in the notes.
Hand-in problems due Tuesday May 19th

Problem 13.2 — evaluating integrals by matching

Find useful approximations to

\[ F(x) \overset{\text{def}}{=} \int_0^\infty \frac{du}{\sqrt{x^2 + u^2 + u^4}} \]  

as (i) \( x \to 0 \); (ii) \( x \to \infty \). (iii) Use MATLAB to compare your approximations with a numerical evaluation of the integral on the range \( 0 < x < 10 \).

Problem 14.2 — basic boundary layers

(i) Solve the boundary value problem

\[ h_x = \epsilon h_{xx} + \sin x, \quad u(0) = u(\pi) = 0, \]

exactly. To assist communication, please use the notation

\[ X \overset{\text{def}}{=} \frac{x - \pi}{\epsilon}, \quad \text{and} \quad E \overset{\text{def}}{=} e^{-\pi/\epsilon}. \]

This should enable you to write the exact solution in a compact form. (ii) Find the first three terms in the regular perturbation expansion of the outer solution

\[ h(x) = u_0(x) + \epsilon h_1(x) + \epsilon^2 h_2(x) + O(\epsilon^3). \]

(iii) There is a boundary layer at \( x = \pi \). “Rescale” the equation using \( X \) above as the independent variable and denote the solution in the boundary layer by \( H(X) \). Find the first three terms in the regular perturbation expansion of the boundary-layer equation:

\[ H = H_0(X) + \epsilon H_1(X) + \epsilon^2 H_2(X) + O(\epsilon^3). \]

(iv) The \( H_n \)’s above will each contain an unknown constant. Determine the three constants by matching to the interior solution. (v) Construct a uniformly valid solution, up to an including terms of order \( \epsilon^2 \). (vi) With \( \epsilon = 0.5 \), use MATLAB to compare the exact solution from part (i) with the approximation in part (v).