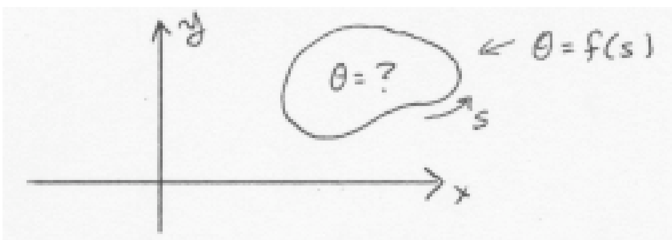
7. Elliptic equations. Variational principles.

Of the 3 canonical forms listed in Section 5, we have yet to discuss Laplace's equation,

(1) $$\theta_{xx} + \theta_{yy} = 0 .$$

This equation relates to many topics, but especially complex variables and the theory of analytic functions; recall that the real and imaginary parts of an analytic function each obey (1). The hallmark of analytic functions is *smoothness*. Indeed, in the previous section we proved that solutions of (1) do not admit discontinuous derivatives.

Pursuing this idea of smoothness we ask: What is the smoothest function defined inside some closed curve on the *x-y* plane that takes the prescribed value *f*(*s*) on the curve?



The answer depends on the definition of smoothness. A natural measure of *roughness* is

(2) $$R[\theta] = \iint \nabla\theta \cdot \nabla\theta \; dx\,dy$$

where the integral is over the region inside the curve. The smoothest $\theta(x,y)$ is that which *minimizes* $R[\theta]$.

A *function* is a number that depends on other numbers. Thus $\theta(x,y)$ is a function of (x,y). A *functional* is a number that depends on the values of a function throughout some range of its arguments. Thus $R[\theta]$ is a functional of $\theta(x,y)$.

The problem of finding the $\theta(x,y)$ that minimizes $R[\theta]$ is *analogous* to the problem of finding the (x,y) at which $\theta(x,y)$ takes its minimum value. Let (x,y) be the sought-for location of the minimum. Expanding

(3) $$\begin{aligned} \theta(x+\delta x, y+\delta y) &= \theta(x,y) + \theta_x(x,y)\delta x + \theta_y(x,y)\delta y \\ &\quad + \frac{1}{2}\theta_{xx}(x,y)\delta x^2 + \theta_{xy}(x,y)\delta x\,\delta y + \frac{1}{2}\theta_{yy}(x,y)\delta y^2 + \cdots \end{aligned}$$

we see that if (x,y) is the location of the minimum, then

(4) $$\theta_x(x,y) = \theta_y(x,y) = 0 ,$$

and the contribution of the quadratic terms in (3) must be positive definite. From our discussion of quadratic forms in Section 5 we know that the quadratic terms in (3) are positive definite if

(5) $\theta_{xx} > 0, \quad \theta_{yy} > 0, \quad \theta_{xx}\theta_{yy} > \theta_{xy}^{\ 2}$.

The conditions (4) would also hold at a maximum of θ, or at an inflection point.

Now consider the analogous problem of minimizing the functional $R[\theta]$. Let $\theta(x, y)$ be the sought-for *function*. Let $\theta(x, y) + \delta\theta(x,y)$ be a nearby function. Both θ and $\theta + \delta\theta$ are equal to f on the boundary. Hence $\delta\theta = 0$ on the boundary. Expanding

(6)
$$\begin{aligned} R[\theta + \delta\theta] &= \iint \nabla(\theta + \delta\theta) \cdot \nabla(\theta + \delta\theta)\, dx\, dy \\ &= \iint \nabla\theta \cdot \nabla\theta + 2\iint \nabla\theta \cdot \nabla\delta\theta + \iint \nabla\delta\theta \cdot \nabla\delta\theta \end{aligned}$$

and noting the analogy between (6) and (3), we see that a *necessary* condition for $\theta(x, y)$ to be a minimum of $R[\theta]$ is

(7) $\iint \nabla\theta \cdot \nabla\delta\theta = 0$.

But $\nabla\theta \cdot \nabla\delta\theta = \nabla \cdot (\delta\theta\nabla\theta) - \delta\theta\nabla^2\theta$, and

(8) $\iint \nabla \cdot (\delta\theta\nabla\theta) = 0$

because $\delta\theta = 0$ on the boundary. Thus (7) implies that

(9) $\iint \delta\theta\, \nabla^2\theta = 0$.

But (9) must hold for *any* $\delta\theta(x, y)$. Thus

(10) $\nabla^2\theta = 0$

within the bounding curve. Thus the function that minimizes $R[\theta]$ subject to the boundary condition θ=f is just the solution of Laplace's equation with the same boundary condition. We see that this solution is a true minimum of $R[\theta]$ because the quadratic terms in (6) are positive definite:

(11) $\iint \nabla\delta\theta \cdot \nabla\delta\theta \ge 0$ for all $\delta\theta(x, y)$.

The differential equation $\nabla^2\theta = 0$ is said to be equivalent to the *variational principle* $\delta R[\theta] = 0$.

What is the variational principle corresponding to the wave equation? Does it represent a minimum? Can you find a variational principle corresponding to the heat equation?

Next consider the functional

(12) $$R_1[\theta] = \iint \left(\frac{1}{2}\nabla\theta\cdot\nabla\theta + \theta F \right)$$

where $F(x,y)$ is a prescribed function. Once again we require that $\delta R_1 = R_1[\theta+\delta\theta] - R_1[\theta]$ vanish at the first order in $\delta\theta$. We continue to assume that both θ and $\theta + \delta\theta$ satisfy the boundary condition $\theta = f$. Hence $\delta\theta = 0$ on the boundary. We find that

(13) $$\delta R_1 = \iint \delta\theta\left(F - \nabla^2\theta \right) + \frac{1}{2}\iint \nabla\delta\theta\cdot\nabla\delta\theta .$$

Thus the solution to $\nabla^2\theta = F$ with boundary condition $\theta = f$ gives R_1 its minimum value. Note that R_1 can be decreased by making θ smoother (first term on the right-hand side of (12)) or by making θ more negatively correlated with F (second term). The solution is a compromise between these 2 things.

Finally we consider

(14) $$R_2[\theta] = \iint \left(\frac{1}{2}\nabla\theta\cdot\nabla\theta + \theta F \right) dx\,dy - \oint g\theta\; ds$$

where s is the coordinate tangent to the boundary, and $g(s)$ is a function prescribed along the boundary. The last integral in (14) makes no contribution unless we relax our assumption that θ and $\theta + \delta\theta$ take the prescribed value f along the boundary. Relaxing this assumption corresponds to making $\delta\theta$ everywhere completely arbitrary. For completely arbitrary $\delta\theta$,

(15) $$\begin{aligned} &\delta R_2 - \frac{1}{2}\iint \left(\nabla\delta\theta\cdot\nabla\delta\theta\right) dx\,dy \\ &\quad= \iint \left\{ \nabla\cdot\left(\delta\theta\,\nabla\theta\right) - \delta\theta\left(\nabla^2\theta - F\right) \right\} dx\,dy - \oint g\,\delta\theta\; ds \\ &\quad= \iint \left(F - \nabla^2\theta \right)\delta\theta\; dx\,dy + \oint \left(\frac{\partial\theta}{\partial n} - g \right) \delta\theta\; ds \end{aligned}$$

Thus, since $\delta\theta$ is arbitrary,

(16) $$\nabla^2\theta = F \qquad \text{with boundary condition} \qquad \frac{\partial\theta}{\partial n} = g .$$

However, we must be careful! If (15) vanishes for arbitrary $\delta\theta$, then it vanishes when $\delta\theta = 1$. But $\delta\theta = 1$ in turn implies that

(17) $$\iint \nabla^2\theta\; dx\,dy - \oint \frac{\partial\theta}{\partial n} ds = \iint F\; dx\,dy - \oint g\; ds$$

(which could aslo be obtained from (16)). The left-hand side of (17) vanishes by the divergence theorem. Therefore the prescribed functions F and g are not in fact completely arbitrary but must satisfy the "consistency relation"

(18) $\iint F\, dx\, dy = \oint g\, ds$.

If (18) is violated then (16) has no solution. If, on the other hand, (18) is satisfied, then (16) has an infinite number of solutions differing from one another in a trivial way. (How?). Astute readers will recognize this as another example of the Fredholm alternative.

We can regard $\delta R_2 = 0$ as the basic variational principle for all the problems discussed so far. If we consider only those θ that satisfy the boundary condition $\theta = f$, then this variational principle implies

(19) $\nabla^2 \theta = F$ with boundary condition $\theta = f$.

[Poisson's equation with Dirichlet boundary conditions.] If we consider θ to be *any function whatsoever* then $\delta R_2 = 0$ implies (16) [Poisson's equation with Neumann boundary conditions] but then F and g must satisfy the consistency relation (18). These 2 problems, (16) and (19), may be interpretted as steady solutions to the 2-dimensional heat equation,

(20) $\theta_t = \kappa\left(\theta_{xx} + \theta_{yy}\right) + Q$

where $Q = -\kappa F$, with boundary condition of prescribed temperature $\theta = f$ or prescribed heat flux $-\kappa\theta_n = -\kappa g$. Then the consistency relation (18) corresponds to the statement that the heating of the interior must be balanced by the heat flux across the boundary.

Variational principles offer a means of solving systems like (16) or (19). Let $\{\phi_1(x,y), \phi_2(x,y), \ldots, \phi_n(x,y)\}$ be a set of n functions. We set

(21) $\theta = \sum_{i=1}^{n} a_i\, \phi_i(x,y)$

where a_i are n coefficients to be determined, and substitute (21) into (14) obtaining

(22) $R_2 = \sum_{i,j} A_{ij} a_i a_j + \sum_i B_i a_i$

where

(23) $A_{ij} = \frac{1}{2} \iint \nabla\phi_i \cdot \nabla\phi_j\, dx\, dy$ and $B_i = \iint \phi_i F\, dx\, dy - \oint g\, \phi_i\, ds$.

Thus R_2 is an ordinary function of the n variables $\{a_1, a_2, \ldots, a_n\}$. Finding the minimum of R_2 corresponds to solving the linear system

(24) $\frac{\partial R_2}{\partial a_k} = 0$ for $k = 1, 2, \ldots, n$.

Since A_{ij} is symmetric we can find a transformation $\bar{a}_i = \bar{a}_i(a_1, a_2, \ldots, a_n)$ that diagonalizes R_2:

(25) $$R_2 = \bar{A}_1\bar{a}_1^{\,2} + \bar{A}_2\bar{a}_2^{\,2} + \cdots + \bar{A}_n\bar{a}_n^{\,2} + \sum_i \bar{B}_i\bar{a}_i \,.$$

Furthermore $\bar{A}_i > 0$ by the minimum property of R_2. In the form (25), the solution is easy:

(26) $$\frac{\partial R_2}{\partial \bar{a}_i} = 2\bar{A}_i\bar{a}_i + \bar{B}_i\bar{a}_i = 0 \qquad \Rightarrow \qquad \bar{a}_i = -\frac{\bar{B}_i}{2\bar{A}_i}.$$

If as $n \to \infty$ the functions $\phi_i(x, y)$ form a complete set, then the solution obtained in this way converges to the unique exact solution.

Example. To solve the trivial system

(27) $$\nabla^2\theta = F, \quad 0 < x, y < \pi \quad \text{with boundary condition } \theta = 0$$

the choice

(28) $$\phi_{nm}(x, y) = \sin(nx)\,\sin(my)$$

is convenient because the functions satisfy the boundary conditions. For this choice of basis functions the matrix A_{ij} is diagonal and the solution is easily obtained. The procedure is clearly equivalent to an eigenfunction expansion.

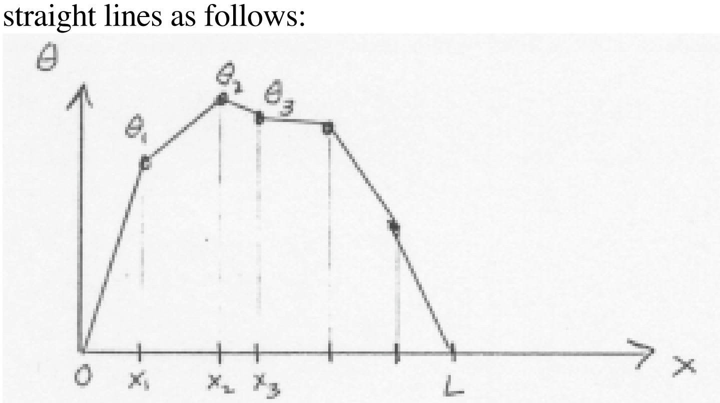
Other choices of basis functions may offer special advantages, especially in irregular domains. Consider first the one-dimensional problem

(29) $$\theta_{xx} = F, \quad 0 < x < L$$

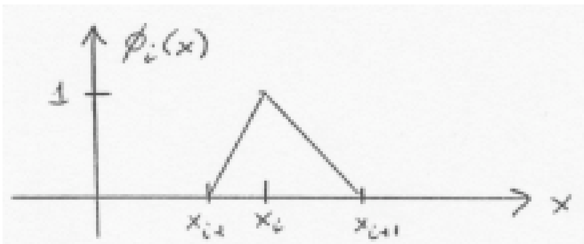
with boundary conditions θ=0. This problem is equivalent to finding the minimum of

(30) $$R = \int dx\left(\frac{1}{2}\theta_x^{\,2} + \theta F\right)$$

with θ pinned to zero at x=0,L. In the method of *finite elements* we represent θ as a series of straight lines as follows:



The *nodal points* x_i are fixed but they need not be evenly spaced; in many practical problems it is desirable to have a greater spatial resolution in one part of the domain than in another. The nodal values θ_i are the new independent variables of the problem. In between the nodal values, θ varies linearly with location as shown on the sketch. This representation is equivalent to $\theta = \sum_i \theta_i \phi_i(x)$ where $\phi_i(x)$ is the "tent function" centered on $i\,\Delta x$:



The integral in (30) may then be carried out, and *R* expressed as an ordinary function of the θ_i. Then the equations

$$(31) \qquad \frac{\partial R}{\partial \theta_i} = 0 \quad \Rightarrow \quad \frac{(\theta_{i+1} - \theta_i)}{(x_{i+1} - x_i)} - \frac{(\theta_i - \theta_{i-1})}{(x_i - x_{i-1})} = \int dx\, \phi_i\, F\,.$$

The integral on the right-hand side could be performed by representing *F* with the same "tent functions" used to represent θ.

Equations (31) represent a system of *n* equations in the *n* nodal values θ_i. Using Taylor expansions one can easily show that (31) are a valid approximation to (30). (In fact, if the nodal points are evenly spaced, then (31) correspond to a standard finite-difference form.)

The beauty of this method is that it extends to higher dimensions and to irregularly shaped boundaries. In 2 dimensions, the standard approach is to cover the domain with triangles or deformed quadrilaterals. Within each such element the dependent variable is some precisely defined interpolate of the nodal values. The functional appearing in the variational principle then becomes an ordinary function of these nodal values, and the equations for the nodal values result from the requirement that the derivatives of this function vanish. This tremendously useful method does not, strictly speaking, require a variational principle, but the existence of a variational principle makes it easy to apply.

Of course it is one thing to write down a system of linear equations for a_i or θ_i; it is quite another matter to actually solve these equations. For large systems it is seldom practical to transform the system into its diagonal form. However, elliptic solvers typically rely on the property that solving the system corresponds to finding the minimum of a function. One of the simplest methods (relaxation) exploits the connection between Laplace's equation and the heat equation.

As in our examples, the boundary conditions for elliptic equations usually take the form of specifying θ or its normal derivative $\partial\theta / \partial n$ (or a mixture of the two) on a closed curve. This is a very different type of boundary condition than the one we applied to the wave equation $\theta_{tt} - \theta_{xx} = 0$. There we prescribed θ *and* its normal derivative θ_t along only *one* of the boundaries, the line t=0. Is such a boundary condition also possible for $\theta_{xx} + \theta_{yy} = 0$?

The answer is *yes*, but it is a *very poor* idea. To see why, we broaden the context and consider the general 2nd-order equation in 2 dimensions:

$$a\theta_{xx} + 2b\theta_{xy} + c\theta_{yy} + \cdots = F \tag{32}$$

We ask: Under whar circumstances is it sensible to prescribe boundary conditions of the form

$$\left.\begin{array}{l} \theta = f \\ \theta_n = g \end{array}\right\} \qquad \text{along some curve } C \text{ in the } xy\text{-plane?} \tag{33}$$

The combination (32-33) is called the *Cauchy problem*. It is the second-order analogue of the problem considered in Section 2 for first-order equations. In Section 2 we concluded that θ could be specified along a curve provided that it was not a characteristic. We shall reach a similar conclusion for (32-33).

Let C be parameterized as

$$x = x(s), \qquad y = y(s) \tag{34}$$

If we know $\theta(s)$ and $\theta_n(s)$, then we know

$$p(s) \equiv \theta_x(s) \qquad \text{and} \qquad q(s) \equiv \theta_y(s). \tag{35}$$

Can we then somehow determine $\theta_{xx}(s)$, $\theta_{xy}(s)$, and $\theta_{yy}(s)$? If we can, then we will know $\theta_{nn}(s)$, and hence we will know θ and θ_n on a curve infinitesimally close to C. Continuing this process, we can determine θ in a region around C.

To determine the second derivatives, we first rewrite the boundary condition (35) as

$$\begin{aligned} \theta_x(x(s), y(s)) &= p(s) \\ \theta_y(x(s), y(s)) &= q(s) \end{aligned} \tag{36}$$

and differentiate along the curve to obtain

$$\begin{aligned} \theta_{xx}\frac{dx}{ds} + \theta_{xy}\frac{dy}{ds} &= p'(s) \\ \theta_{yx}\frac{dx}{ds} + \theta_{yy}\frac{dy}{ds} &= q'(s) \end{aligned} \tag{37}$$

The right-hand sides of (36) and (37) are known functions. Eqns (37) are 2 equations in the 3 unknowns θ_{xx}, θ_{xy}, and θ_{yy}; (32) itself provides the 3rd equation. The complete system

$$(38)\qquad \begin{bmatrix} dx/ds & dy/ds & 0 \\ 0 & dx/ds & dy/ds \\ a & 2b & c \end{bmatrix}\begin{bmatrix}\theta_{xx} \\ \theta_{xy} \\ \theta_{yy}\end{bmatrix} = known$$

has a unique solution provided that the determinant does not vanish, that is, provided

$$(39)\qquad a\left(\frac{dy}{ds}\right)^2 - 2b\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) + c\left(\frac{dx}{ds}\right)^2 \neq 0.$$

In other words, the Cauchy problem is solvable provided that C is not tangent to either of the lines defined by

$$(40)\qquad \frac{dy}{dx} = \frac{b}{a} \pm \frac{\sqrt{b^2 - ac}}{a}$$

If we agree to call these lines characteristics, then we see that this definition agrees with our use of the term in the case of the wave equation. (There we specified Cauchy data along the line t=0, which does not coincide with either of the lines $x \pm c\,t = const$.) In the elliptic case ($b^2 - 4ac < 0$) there are no real characteristics, so it is impossible to disobey the rule. Nevertheless Cauchy boundary conditions on an elliptic equation produce bizarre solutions.

This is illustrated by a famous example discovered by Hadamard. The problem is

$$(41)\qquad \theta_{xx} + \theta_{yy} = 0$$

with boundary conditions

$$(42)\qquad \theta = 0 \qquad \text{and} \qquad \theta_x = \alpha\,\sin ky \qquad \text{on } x=0,$$

where α and k are constants. The solution is

$$(43)\qquad \theta(x,y) = \frac{\alpha}{k}\sinh kx\,\sin ky$$

which blows up as $x \to \pm\infty$. However the blow-up itself is not offensive. To see what is, suppose that $\alpha = 1/k$. Then as $k \to \infty$ the boundary conditions smoothly approach $\theta = \theta_x = 0$, while at any nonzero x the solution becomes very large. It is this sensitivity to the Cauchy boundary conditions that offended Hadamard and led him to his definition of a well-posed problem:

1.) The solution must exist.
2.) It must be unique.
3.) It *cannot* depend sensitively on the boundary conditions.

We end this section with a discussion of the equation,

$$(44)\qquad \nabla^2\theta + \lambda\theta = 0$$

with boundary condition θ=*f*. When λ>0 (44) is called the Helmholtz equation; when λ<0 it is called *modified* Helmholtz. Some of the importance of (44) comes from the fact that any equation of the form

(45) $$\theta_{xx} + \theta_{yy} + a\theta_x + b\theta_y + c\theta = 0$$

can be transformed into the form (44) by a change in the dependent variable. (How?)

For either sign λ (44) is equivalent to the variational principle,

(46) $$\delta \iint \left[\nabla\theta \cdot \nabla\theta - \lambda\theta^2 \right] dx\, dy = 0 .$$

However, only in the case λ<0 are the terms positive definite so that the stationary value is a true minimum. When λ<0 the solution is unique. To see this let $u(x,y)$ be the difference between any 2 solutions. Then u satisfies

(47) $\nabla^2 u + \lambda u = 0$ and the boundary condition u=0.

Multiplying (47) by u and integrating, we obtain

(48) $$\delta \iint \left[\nabla u \cdot \nabla u - \lambda u^2 \right] dx\, dy = 0$$

which implies u=0 if λ<0. This proof clearly depends on the minimum property.

When λ>0, the solution of (44) is not necessarily unique. For particular negative values of λ (eigenvalues) (47) has nonvanishing solutions (eigensolutions). Thus different signs of λ imply huge differences in behavior. The lower-order derivatives really matter!

These differences in behavior are physically understandable if one regards (44) as the steady solution of the heat equation with an additional term,

(49) $$\frac{\partial\theta}{\partial t} = \nabla^2\theta + \lambda\theta .$$

When λ<0 both terms on the right-hand side of (49) cause decay. However, when λ>0 the last term causes growth. From this point of view the eigenfunctions represent a sensitive balance between growth and decay.

From still another viewpoint, the case λ>0 corresponds to forced-wave solutions of

(50) $$\theta_{tt} = c^2\nabla^2\theta$$

If $\theta(x,y,t) = e^{-i\omega t}\theta(x,y)$ then

(51) $$-\omega^2\theta = c^2\nabla^2\theta$$

which fits the form of (44) with $\lambda = \omega^2 / c^2 > 0$. From this viewpoint, the Helmholtz equation, although technically elliptic, is just the wave equation in disguise!

Reference. Zauderer chapter 8. Gustafson.