9. Higher order PDEs as systems of first-order PDEs. Hyperbolic systems.

For PDEs, as for ODEs, we may reduce the order by defining new dependent variables. For example, in the case of the wave equation,

(1) $$\theta_{tt} = c^2\theta_{xx},$$

the definitions

(2) $$u \equiv \theta_t \quad \text{and} \quad v \equiv \theta_x$$

imply

(3) $$u_x = v_t,$$

while (10) itself may be written as $u_t = c^2 v_x$. Thus the second-order *equation* (1) is equivalent to the first-order *system*

(4) $$\begin{aligned} u_t - c^2 v_x &= 0 \\ u_x - v_t &= 0 \end{aligned}$$

The motivation for this approach is our success with first-order equations. We found that

(5) $$a(x,t)\frac{\partial\theta}{\partial t} + b(x,t)\frac{\partial\theta}{\partial x} = f(x,t)$$

could be written as

(6) $$\frac{d\theta}{ds} = f$$

where

(7) $$\frac{d}{ds} = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}$$

is the directional derivative along the curve

(8) $$\frac{dt}{ds} = a, \qquad \frac{dx}{ds} = b.$$

Can we do a similar trick for first-order *systems*?

Consider the general system with 2 dependent variables, $u(t,x)$ and $v(t,x)$, in the 2 independent variables x and t:

(9) $$\begin{aligned} D_1 u + d_1 v &= f_1 \\ D_2 u + d_2 v &= f_2 \end{aligned}$$

Here,

$$
(10)\qquad \begin{aligned} D_1 &= A_1(x,t,u,v)\frac{\partial}{\partial t} + B_1(x,t,u,v)\frac{\partial}{\partial x} \\ d_1 &= a_1(x,t,u,v)\frac{\partial}{\partial t} + b_1(x,t,u,v)\frac{\partial}{\partial x} \end{aligned}
$$

(and similarly, with 2 replacing 1) are the directional derivatives that appear in the given system. We allow the coefficients to depend on *u* and *v* as well as *x* and *t*; such systems are called *quasilinear*.

The system (9) is the generalization of (5) to the case of 2 dependent variables. The theory can be extended to *n* dependent variables in *m* independent variables (see Whitham chapter 5). However, success is rare when there are more than 2 *independent* variables, and more than 2 *dependent* variables complicates the notation; for these reasons we shall be content with (9).

The primary difference between (5) and (9) is that each equation in (9) contains 2 directional derivatives which generally "point" in different directions. Thus it is generally impossible to integrate either of (9) in the same way as (5). But what about linear combinations of (9)? Multiplying (9a) by c_1 and (9b) by c_2, where c_1 and c_2 are functions of (*x,t,u,v*) to be determined, and adding the equations, we obtain

$$
(11)\qquad (c_1D_1 + c_2D_2)u + (c_1d_1 + c_2d_2)v = c_1f_1 + c_2f_2 .
$$

We want the directional derivative of *u* in (11) to be proportional to the directional derivative of *v* in the same equation. That is, we want

$$
(12)\qquad c_1D_1 + c_2D_2 = \alpha(c_1d_1 + c_2d_2)
$$

where α (another function of (x,t,u,v)) is the factor of proportionality. Rewriting (12) in the form

$$
(12a)\qquad (\cdots)\frac{\partial}{\partial t} + (\cdots)\frac{\partial}{\partial x} = 0
$$

and requiring the coefficients of $\partial / \partial t$ and $\partial / \partial x$ to vanish, we obtain 2 equations for c_1 and c_2:

$$
(13)\qquad \begin{pmatrix} A_1 - \alpha a_1 & A_2 - \alpha a_2 \\ B_1 - \alpha b_1 & B_2 - \alpha b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

For nonzero (c_1,c_2), the determinant in (13) must vanish. This gives a quadratic equation for α. If the 2 roots of this quadratic equation are real, and if the corresponding pairs (c_1,c_2) are different (so that the 2 equations of form (11) are independent equations), then we have achieved our goal of writing (9) in the form of 2 equations, each of which involves derivatives along a single family of lines in the *xt*-plane. In this case, the system is said to be *hyperbolic*, and the 2 families of lines are its characteristics. If we are very lucky, we might even succeed in manipulating (11) into the form

$$\text{(14)} \qquad \frac{dR_1}{ds_1} = F_1, \qquad \frac{dR_2}{ds_2} = F_2$$

where d/ds_1 and d/ds_2 are the directional derivatives corresponding to the 2 families of characteristics. Then, if it happens that $F_1 = F_2 = 0$, the variables $R_1(t,x,u,v)$ and $R_2(t,x,u,v)$ are called *Riemann invariants*. In exceptional cases, the 2 families of characteristics may actually coincide.

In working out particular problems, it is almost never worthwhile to use the general notation used above. It is almost always better to simply follow the foregoing strategy in each particular case. However, the general notation shows why the method usually fails in the case of 3 or more independent variables. In the case of 2 dependent variables in 3 independent variables, (13) becomes a set of 3 equations (corresponding to the 3 kinds of first-derivative) in the 2 unknowns (c_1,c_2) and is thus generally overdetermined. In the remainder of this section, we focus on examples.

Example. In the case of system (4) the linear combination is

$$c_1(u_x - v_t) + c_2(u_t - c^2 v_x) = 0$$

so we want

$$(c_2, c_1) = \alpha(-c_1, -c^2 c_2), \qquad \text{that is,} \qquad \begin{pmatrix} \alpha & 1 \\ 1 & \alpha c^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $\alpha^2 c^2 - 1 = 0 \quad \Rightarrow \quad \alpha = \pm 1/c$. The choice $\alpha = +1/c$ corresponds to $d/ds_1 = \partial_t - c\partial_x$ and $R_1 = u + cv$. The choice $\alpha = -1/c$ corresponds to $d/ds_2 = \partial_t + c\partial_x$ and $R_2 = u - cv$.

Example. For the heat equation, $\theta_t - \kappa\theta_{xx} = 0$, we let $u = \theta$ and $v = \theta_x$ to obtain the system

$$u_t - \kappa v_x = 0$$
$$v - u_x = 0$$

The second equation fits the desired form, but no other linear combination does, because there is no v_t-term. Thus there is only one characteristic, and therefore the system is *not* hyperbolic (big surprise).

Example. For Laplace's equation, $\theta_{xx} + \theta_{yy} = 0$, the choices $u = \theta_x$ and $v = \theta_y$ yield the system

$$u_x + v_y = 0$$
$$u_y - v_x = 0$$

The general linear combination is

$$c_1(u_x + v_y) + c_2(u_y - v_x) = (c_1\partial_x + c_2\partial_y)u + (c_1\partial_y - c_2\partial_x)v = 0 .$$

Thus we want

$$(c_1\partial_x + c_2\partial_y) = \alpha(c_1\partial_y - c_2\partial_x) \quad \Rightarrow \quad \begin{pmatrix} 1 & \alpha \\ \alpha & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \alpha = \pm i .$$

Thus there are no characteristics, and the system is not hyperbolic.

Example.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - e^t\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + v\right) = 0$$
$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + x\left(\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x}\right) - v = 0$$

These equations are *already* in characteristic form; the directional derivatives in each equation point in the same direction. Defining

$$\frac{d}{ds} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \qquad \text{and} \qquad \frac{d}{dr} = \frac{\partial}{\partial t} - \frac{\partial}{\partial x}$$

we have

$$(*) \qquad \frac{du}{ds} - e^t\left(\frac{dv}{ds} + v\right) = 0 \qquad \text{and} \qquad \frac{du}{dr} + x\frac{dv}{dr} - v = 0 .$$

The first of these implies

$$\frac{du}{ds} - \frac{d}{ds}(e^t v) + v\frac{d}{ds}e^t - e^t v = 0 \qquad \Rightarrow \qquad \frac{d}{ds}(u - e^t v) = 0$$

since $dt/ds = 1$. Similarly, the second of (*) implies

$$\frac{d}{dr}(u + xv) - v\frac{dx}{dr} - v = 0 \qquad \Rightarrow \qquad \frac{d}{dr}(u + xv) = 0$$

since $dx/dr = -1$. Thus the 2 Riemann invariants are $u - e^t v$ and $u + xv$. The characteristics corresponding to d/ds are $x - t = const$. Thus $u - e^t v = F(x - t)$ where F is an arbitrary function. Similarly, the characteristics corresponding to d/dr are

$x+t=const$. Thus $u+xv=G(x+t)$ where G is another arbitrary function. The general solution can be written

$$u=\frac{x\,F(x-t)+e^t\,G(x+t)}{x+e^t},\qquad v=\frac{G(x+t)-F(x-t)}{x+e^t}$$

In the remainder of this section we consider a far more interesting example, the one-dimensional shallow-water equations:

$$(15)\qquad \begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial h}{\partial x}\\ &\frac{\partial h}{\partial t}+h\frac{\partial u}{\partial x}+u\frac{\partial h}{\partial x}=0\end{aligned}$$

Note that these equations are nonlinear.

Before proceeding with the analysis of (15), we note that these equations may be viewed as a special case of the equations for a one-dimensional polytrope, a homentropic gas with equation of state $p=c\rho^\gamma$, where c and γ are constants. The equations governing the gas are

$$u_t+uu_x=-p_x/\rho \qquad \text{and} \qquad \rho_t+u\rho_x+\rho u_x=0.$$

Thus (15) correspond to $\rho=h$, $c=g/2$ and $\gamma=2$. Using this analogy, many of the general results from gas dynamics (Whitham chapter 6) may be taken over to the shallow-water system.

The general linear combination of (15) is

$$(16)\qquad c_1\left(u_t+u\,u_x+g\,h_x\right)+c_2\left(h_t+h\,u_x+u\,h_x\right)=0$$

which is equivalent to

$$(17)\qquad \frac{d}{ds_1}u+\frac{d}{ds_2}h=0$$

where

$$(18)\qquad \frac{d}{ds_1}\equiv c_1\frac{\partial}{\partial t}+c_1\,u\frac{\partial}{\partial x}+c_2\,h\frac{\partial}{\partial x}$$

and

$$(19)\qquad \frac{d}{ds_2}\equiv c_1\,g\frac{\partial}{\partial x}+c_2\frac{\partial}{\partial t}+c_2\,u\frac{\partial}{\partial x}.$$

To make the directional derivatives proportional we set

$$\frac{d}{ds_1} = \alpha \frac{d}{ds_2} \tag{20}$$

which implies

$$\begin{aligned} c_1 - \alpha c_2 &= 0 \\ c_1 u + c_2 h - \alpha g c_1 - \alpha c_2 u &= 0 \end{aligned} \tag{21}$$

Substituting (21a) into (21b) we obtain

$$c_2 h - \alpha^2 g c_2 = 0 . \tag{22}$$

If $c_2 \neq 0$ then $\alpha^2 = h / g$ and we obtain $\alpha = \pm\sqrt{h / g}$. Thus the system is hyperpolic if h>0. We choose $c_2 = g$; then $c_1 = \pm\sqrt{gh}$; and we have

$$\frac{d}{ds_2} = g\left(\partial_t + \left(u \pm \sqrt{gh}\right)\partial_x \right) \equiv g\frac{d}{ds} \tag{23}$$

and

$$\frac{d}{ds_1} = \alpha \frac{d}{ds_2} = \pm\sqrt{gh}\,\frac{d}{ds} . \tag{24}$$

Thus (17) becomes

$$\pm\sqrt{gh}\frac{du}{ds} + g\frac{dh}{ds} = 0 \qquad \Rightarrow \qquad \frac{d}{ds}\left(u \pm 2\sqrt{gh}\right) = 0 \qquad \Rightarrow \tag{25}$$

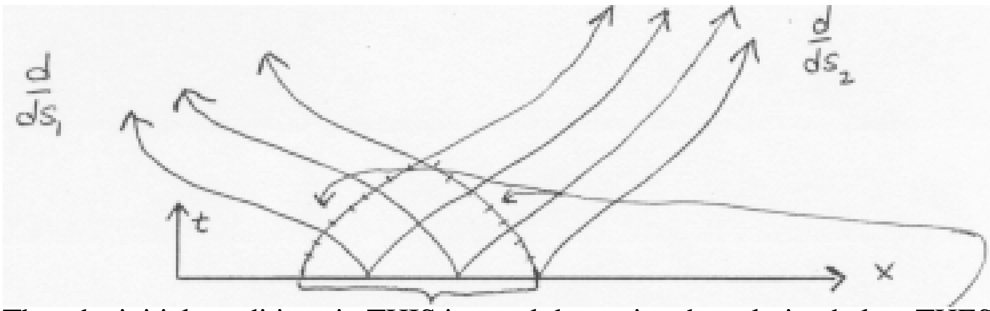
$$\left[\frac{\partial}{\partial t} + \left(u \pm \sqrt{gh}\right)\frac{\partial}{\partial x}\right]\left(u \pm 2\sqrt{gh}\right) = 0 \tag{26}$$

where both signs are to be taken the same. The 2 equations (26) fit the form of (14) with

$$\begin{aligned} &R_1 = u + 2\sqrt{gh} && R_2 = u - 2\sqrt{gh} \\ &\frac{d}{ds_1} = \frac{\partial}{\partial t} + \left(u + \sqrt{gh}\right)\frac{\partial}{\partial x} && \frac{d}{ds_1} = \frac{\partial}{\partial t} + \left(u - \sqrt{gh}\right)\frac{\partial}{\partial x} \end{aligned} \tag{27}$$

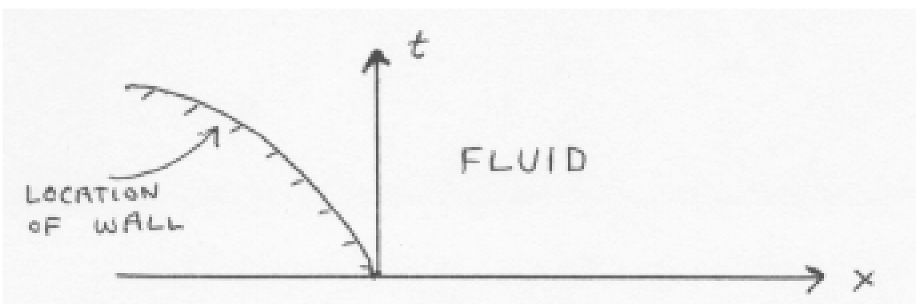
and $F_1 = F_2 = 0$ (where we have re-defined d / ds_1 and d / ds_2).

Suppose that u and h are given along some curve in the *x-t* plane, such as t=0. Then the solution elsewhere is obtained by integrating along the curves defined by the directional derivatives d / ds_1 and d / ds_2. Since these directional derivatives depend on both u and h, each step in this integration requires information carried by *both* sets of characteristics:



Thus the initial conditions in THIS interval determine the solution below THESE 2 characteristics. This situation differs from the linear equation $\theta_{tt} = c^2\theta_{xx}$ in that the characteristics have a slope that depends on u and h. Just as in the case of quasilinear *first*-order equations, this may lead to multi-valued solutions. For general hyperbolic systems the situation can become very complex.

We consider the following initial-value problem for our shallow-water system: The fluid is initially at rest (u=0) with uniform depth ($h = h_0$), and lies to the right of a wall at x=0. Beginning at t=0, the wall moves to the left along the trajectory x=$X(t)$ as shown:



What is the motion of the fluid?

From the analysis above we know that $u + 2\sqrt{gh}$ is constant along (+) characteristics with slope $dx/dt = u + \sqrt{gh}$, while $u - 2\sqrt{gh}$ is constant along (-) characteristics with slope $dx/dt = u - \sqrt{gh}$.

If $u < \sqrt{gh}$, then the (-) characteristics originating on the positive x-axis fill up the entire area of the x-t plane occupied by the fluid. This implies that

(28) $$u - 2\sqrt{gh} = -2\sqrt{gh_0} \qquad \text{throughout the fluid.}$$

On the (+) characteristics with slope

(29) $$\frac{dx}{dt} = u + \sqrt{gh}$$

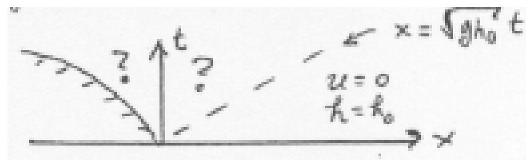
we have

(30) $$u + 2\sqrt{gh} = const,$$

but the constant in (30) is different along each particular characteristic. Nevertheless it follows from (30) *and* (28) that u and h are (different) constants along each (+) characteristic. Hence by (29) each of the (+) characteristics is a straight line.

For the (+) characteristics intersecting the x-axis, the slope is $dx/dt = \sqrt{gh_0}$. On these characteristics the constant in (30) is $2\sqrt{gh_0}$. Thus we conclude that

(31) $$\left.\begin{aligned} u &= 0 \\ h &= h_0 \end{aligned}\right\} \qquad \text{for all} \quad x > \sqrt{gh_0}\, t.$$



To determine the solution in the remaining part of the domain, we use the fact that $u = \dot{X}(t)$ at the wall.

By (28) and (29) the slope of the (+) characteristics may be written as

(32) $$\frac{dx}{dt} = \frac{3}{2}u + \sqrt{gh_0}$$

Thus

(33) $$\frac{dx}{dt} = \frac{3}{2}\dot{X}(\tau) + \sqrt{gh_0}$$

where τ is the time at which the characteristic intersects the wall. Since, as we already know, the right-hand side of (33) must be a constant, it follows from (33) that

(34) $$x = X(\tau) + \left(\frac{3}{2}\dot{X}(\tau) + \sqrt{gh_0}\right)(t - \tau).$$

Given any (x,t) in the region of the moving fluid, we can determine $u(x,t)$ and $h(x,t)$ as follows. First we solve (34) for τ, the time at which the (+) characteristic passing through (x,t) intersects the wall. Then

(35) $$u(x,t) = \dot{X}(\tau)$$

and by (28)

(36) $$\sqrt{gh(x,t)} = \sqrt{gh_0} + \frac{1}{2}\dot{X}(\tau).$$

Eqns (35) and (36) hold at all points along the characteristic. The solution will be single-valued if (34) has only one solution for τ.

An interesting special case is

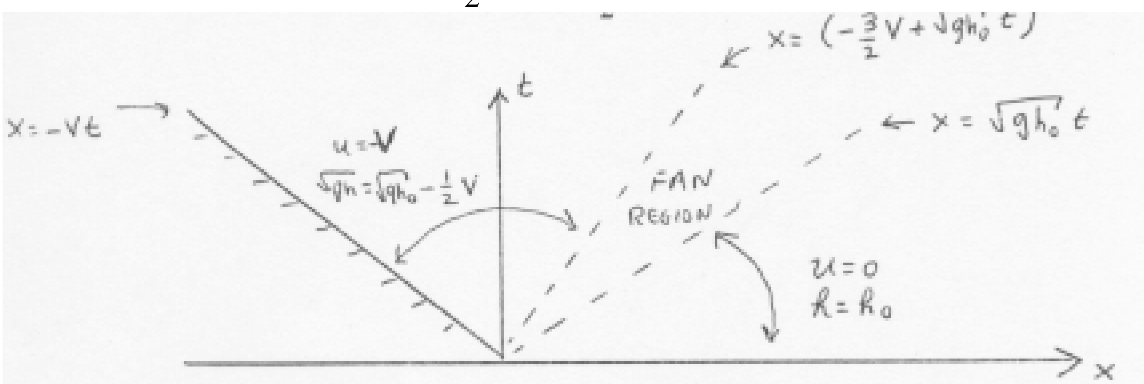
(37) $\dot{X} = -V$ (constant)

corresponding to a wall moving steadily to the left at speed V>0. In this case the slope of the (+) characteristics intersecting the wall is

(38) $$\frac{dx}{dt} = -\frac{3}{2}V + \sqrt{gh_0}$$

On these characteristics the solution is

(39) $$u = -V, \qquad \sqrt{gh} = \sqrt{gh_0} - \frac{1}{2}V$$



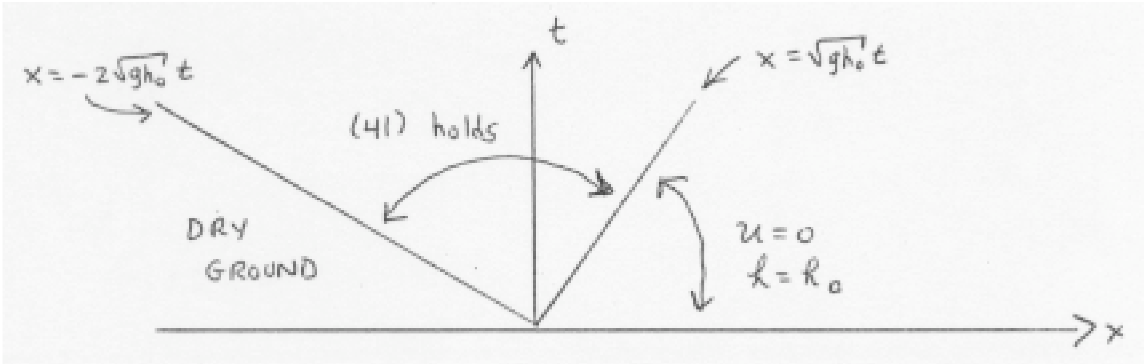
Between this region and the region of quiescent flow lies a fan region, in which the (+) characteristics pass through the origin with every slope between (38) and $\sqrt{gh_0}$, and the solution varies smoothly across the characteristics. As $V \to 2\sqrt{gh_0}$, the fan extends right up to the wall, and the depth at the wall vanishes. When $V > 2\sqrt{gh_0}$ the wall simply outruns the fluid, and the problem reduces to the "dam break" problem, in which the wall is made to disappear at t=0.

In the dam-break problem, all the (+) characteristics to the left of the quiescent region on $x > \sqrt{gh_0}\, t$ emanate from the origin $x = t = 0$. Each (+) characteristic has the constant slope (29), and since it passes through the origin, we may write its equation as

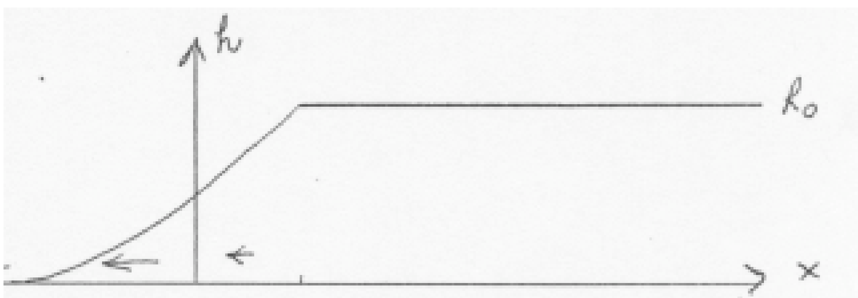
(40) $$x = \left(u + \sqrt{gh}\right)t\,.$$

But then the solution in the entire fan region may be obtained from (40) and (28). We find

(41) $$u = \frac{2}{3}\left(\frac{x}{t} - \sqrt{gh_0}\right) \qquad \text{and} \qquad \sqrt{gh} = \frac{2}{3}\sqrt{gh_0} + \frac{1}{3}\frac{x}{t}$$



Once again, the rightward edge of the fan region is $x = \sqrt{gh_0}\,t$. At this edge u=0 and $h = h_0$, matching the solution on $x > \sqrt{gh_0}\,t$. At the leftward edge, $x = -2\sqrt{gh_0}\,t$, h=0, and $u = -2\sqrt{gh_0}$. This is the speed at which the flood covers dry ground. The solution looks like this:



Cases, like that just considered, in which one of the Riemann invariants is uniform throughout the flow, are called *simple waves*. In such cases one can usually use the uniformity of the Riemann invariant to eliminate one of the dependent variables a priori, and thereby reduce the entire problem to a first-order equation. For example, in the dam-break problem we may use (28) to eliminate u in

$$(42) \qquad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

which then takes the form

$$(43) \qquad \frac{\partial h}{\partial t} + c(h)\frac{\partial h}{\partial x} = 0 \qquad \text{with} \qquad c(h) = \left(3\sqrt{gh} - 2\sqrt{gh_0}\right).$$

It is in fact far simpler to solve the dam break problem in the form (43) than by the previous method; it follows easily from (43) that

$$(44) \qquad \sqrt{gh} = \begin{cases} \sqrt{gh_0}, & x > \sqrt{gh_0}t \\ \frac{2}{3}\sqrt{gh_0} + \frac{1}{3}\frac{x}{t}, & -2\sqrt{gh_0}t < x < \sqrt{gh_0}t \end{cases}$$

and u may then be obtained from (28).

Now, returning to the problem with the moving wall, we consider the case $X = Vt$ in which the wall moves to the *right* at a uniform speed V. We shall find that the fan appearing in the previous solution is replaced by a shock.

We begin by noting, from symmetry considerations, that, just as in the case of the leftward moving wall, the solution must take the form $u = u(x/t)$, $h = h(x/t)$. The boundary conditions are u=0 and $h = h_0$ along the x-axis, and u=V along $x = Vt$, the trajectory of the wall. We let $h = h_w$ along the wall trajectory, where h_w is a constant which must be determined by our solution to the problem.

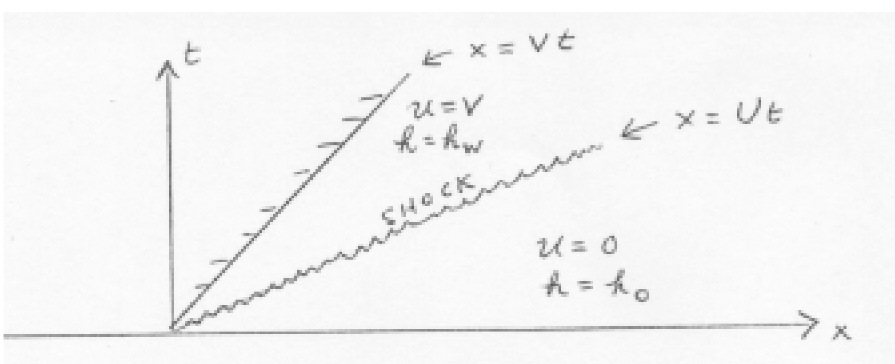
Let (x,t) be a point in the fluid domain. If the 2 characteristics through (x,t) intersect the x-axis, then, as before,

$$\begin{aligned} u + 2\sqrt{gh} &= 0 + 2\sqrt{gh_0} \\ u - 2\sqrt{gh} &= 0 - 2\sqrt{gh_0} \end{aligned} \tag{45}$$

imply that $u = 0$ and $h = h_0$; the fluid is quiescent as in its initial state. On the other hand, if the 2 characteristics through (x,t) intersect the wall trajectory, then

$$\begin{aligned} u + 2\sqrt{gh} &= V + 2\sqrt{gh_w} \\ u - 2\sqrt{gh} &= V - 2\sqrt{gh_w} \end{aligned} \tag{46}$$

imply that $u = V$ and $h = h_w$; the fluid moves at the same speed as the wall, but its uniform depth h_w is as yet undetermined. These 2 solutions obviously disagree, and the only conclusion can be that the quiescent solution holds in a wedge-shaped region near the x-axis, whereas the solution with $u = V$ holds in a wedge-shaped region near the line $x = Vt$. These 2 regions are separated by a *shock* along $x = Ut$, where U too must be determined.



To analyze the shock with complete physical correctness, we must add viscosity to the problem, and treat the shock as an internal boundary layer in which u and h vary rapidly between the uniform values on either side. However, as we realize from our study of

Burger's equation, we can model the shock as a true discontinuity if we are careful to apply jump conditions that express conservation laws that would survive the inclusion of viscosity.

The shallow-water equations imply the conservation of mass,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \tag{47}$$

and momentum

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2 + hu^2\right) = 0\,. \tag{48}$$

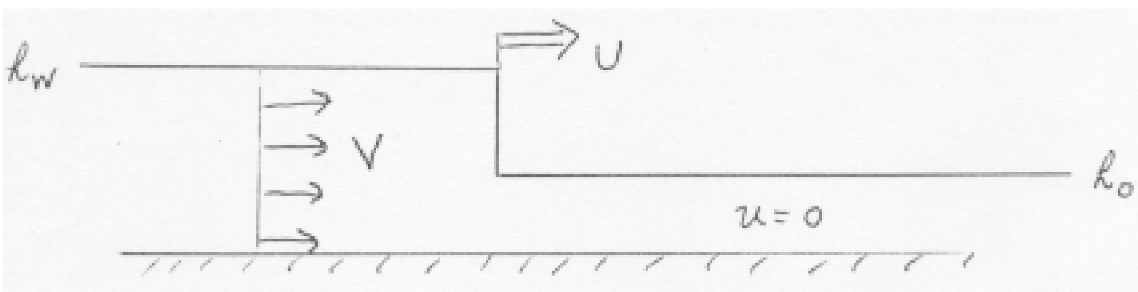
Both of these fit the form

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{49}$$

considered in Section 4. There we showed that (49) implies

$$U[P] = [Q] \tag{50}$$

where U is the velocity of the shock, and [ ] denotes the jump across the shock.



In the present case the jump conditions imply

$$U(h_w - h_0) = Vh_w - 0 \tag{51}$$

and

$$U(Vh_w - 0) = \left(\frac{1}{2}gh_w^{\ 2} + h_w V^2\right) - \frac{1}{2}gh_0^{\ 2}. \tag{52}$$

Eqns (51) and (52) determine the values of h_w and U, completing the solution. We find that

$$U = \sqrt{\frac{gh_w}{2h_0}(h_0 + h_w)} \qquad \text{and} \qquad V = U\frac{(h_w - h_0)}{h_w}. \tag{53}$$

(It is easier to regard U and V as functions of h_w.)

Reference. Whitham chapters 5 & 6 and pp. 454-460.