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# New equations for nearly geostrophic flow

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I have used a novel approach based upon Hamiltonian mechanics to derive new equations for nearly geostrophic motion in a shallow homogeneous fluid. The equations have the same order accuracy as (say) the quasigeostrophic equations, but they allow order-one variations in the depth and Coriolis parameter. My equations exactly conserve proper analogues of the energy and potential vorticity, and they take a simple form in transformed coordinates.

## 1. Introduction

In a recent paper I derived a new set of approximate equations for nearly geostrophic flow in a shallow layer of homogeneous fluid (Salmon 1983, §4, hereinafter referred to as S83). These equations are noteworthy in that they exactly conserve proper analogues of the total energy and the potential vorticity on fluid particles. The conservation laws were automatically obtained because I applied my approximations directly to the Lagrangian of the fluid, taking care not to break the time and particle-label symmetries associated with the conservation of energy and potential vorticity. My equations are (I believe) the only currently known equations for nearly geostrophic flow that have proper conservation laws, apply to nearly geostrophic flow on all horizontal lengthscales, and accommodate order-one variations in the fluid depth and Coriolis parameter.

This paper has two objectives. The first is to demonstrate the close connection between my equations and the 'semigeostrophic' equations of Hoskins (1975). The semigeostrophic equations, which have been widely used in meteorology, also conserve analogues of the energy and potential vorticity, but only in the case of a *constant* Coriolis parameter. The semigeostrophic equations take a very simple form in cleverly chosen 'geostrophic coordinates'.

My second objective is to present new equations for nearly geostrophic flow with horizontal lengthscales larger than the Rossby deformation radius. These new equations are hardly more complicated than the purely geostrophic 'type 2' equations of Phillips (1963). However, they consistently include the effects of relative vorticity on the large-scale flow. These new equations are therefore the appropriate equations for simple numerical models of the ocean thermocline, in which the deformation radius is barely resolved, but in which inertial boundary layers may be important.

This paper is self-contained, but it should be read as a sequel to S83. Section 2 summarizes the results of S83. Section 3 derives generalized semigeostrophic equations, which possess consistent conservation laws in the case of a *non-constant* Coriolis parameter. The generalized semigeostrophic equations have a Hamiltonian formulation, which is the same as for the S83 equations, to within the accuracy of either approximation. The 'geostrophic coordinates' found by Hoskins turn out to be canonical coordinates. Section 4 derives the new equations for large-scale flow. For simplicity, I focus on the case of a single shallow layer of homogeneous fluid that is horizontally unbounded and quiescent at infinity. However, my methods and results should easily extend to other cases. These will be the subject of future publications.

The best-known approximate equations for nearly geostrophic flow are the 'quasigeostrophic' equations (see e.g. Pedlosky 1979). The quasigeostrophic equations are mathematically simple, and they conserve analogues of the energy and potential vorticity. However, the quasigeostrophic equations do not allow order-one variations in the Coriolis parameter, and hence are inapplicable to planetary-scale flow. Furthermore, in the quasigeostrophic equations, the average density stratification (or the fluid depth, in the presently considered case of a homogeneous fluid) is *prescribed*, and the equations apply only to slight departures from the prescribed state. For these reasons, the quasigeostrophic equations are inferior to any of the approximations discussed in this paper. Of course *all* of these approximations filter out the relatively fast inertia-gravity waves that can make numerical integrations of the primitive equations very costly.

Sophisticated approximation methods based upon a Hamiltonian formulation have been widely used for the study of *integrable* dynamical systems. However, the general equations for fluid motion are almost certainly non-integrable. The approximation methods presented here are simple and direct, and are not intended to produce analytical solutions. My goals are accurate conserving equations that are free of artificial restrictions. I emphasize that the accuracy and conservation properties of all my final results can be verified by pedestrian algebraic calculations. These calculations are often quite lengthy, but they provide an independent check on the results of the Hamiltonian methods.

## 2. The $L_1$ dynamics

Hamilton's principle for a mechanical system with $N$ degrees of freedom can be written in the familiar form

$$\delta \int \mathrm{d}\tau \left\{ \sum_i p_i \frac{\mathrm{d}q_i}{\mathrm{d}\tau} - H(q_1, p_1, \ldots, q_N, p_N) \right\} = 0, \tag{2.1}$$

where $q_i$ are the generalized coordinates, $p_i$ the corresponding momenta, $H$ the Hamiltonian, and $\delta$ corresponds to arbitrary independent variations

$$\delta q_i(\tau), \quad \delta p_i(\tau),$$

at fixed time $\tau$.

The equations governing a shallow rotating layer of inviscid homogeneous fluid are

$$\left.\begin{aligned} \frac{\mathrm{D}u}{\mathrm{D}t} - fv &= -g\frac{\partial h}{\partial x}, \\ \frac{\mathrm{D}v}{\mathrm{D}t} + fu &= -g\frac{\partial h}{\partial y}, \\ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh) &= 0, \end{aligned}\right\} \tag{2.2}$$

where $\boldsymbol{x} = (x, y)$ are horizontal Cartesian coordinates, $\boldsymbol{u} = (u, v)$ the corresponding horizontal velocities, $t$ is the time, $g$ is gravity, $f(x, y)$ is the Coriolis parameter, $h(x, y, t)$

is the depth of the fluid, and $\mathrm{D}/\mathrm{D}t = \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y$. None of the terms in (2.2) has $z$-dependence.

As fully explained in S83, the shallow-water dynamics (2.2) can be expressed in a form analogous to (2.1). Again, let the positions

$$x(a,b,\tau), \quad y(a,b,\tau) \tag{2.3}$$

of marked fluid particles be considered as functions of curvilinear labelling coordinates $(a,b)$ and the time $\tau$. The labelling coordinates, which are analogous to the subscripts in (2.1), remain constant following the columnar motion of the fluid particles. Thus $\partial/\partial\tau \equiv \mathrm{D}/\mathrm{D}t$. It is convenient to assign these labelling coordinates so that

$$\mathrm{d}a\,\mathrm{d}b = \frac{\mathrm{d(mass)}}{\rho} = h\,\mathrm{d}x\,\mathrm{d}y, \tag{2.4}$$

i.e.

$$h = \frac{\partial(a,b)}{\partial(x,y)}, \tag{2.5}$$

where $\rho$ is the constant fluid density. The continuity equation (2.2*c*) is obtained by direct application of $\partial/\partial\tau$ to (2.5). Thus mass conservation is implicit in the particle representation (2.3). As shown in S83, the form of Hamilton's principle analogous to (2.1) and equivalent to (2.2) is

$$\delta \int L\,\mathrm{d}\tau = 0, \tag{2.6}$$

where

$$L = \iint \mathrm{d}a\,\mathrm{d}b\left[(u-R)\frac{\partial x}{\partial\tau} + (v+P)\frac{\partial y}{\partial\tau}\right] - H \tag{2.7}$$

and

$$H = \frac{1}{2}\iint \mathrm{d}a\,\mathrm{d}b\,[u^2+v^2+gh]. \tag{2.8}$$

Here $R(x,y)$ and $P(x,y)$ are any two prescribed functions that satisfy

$$\frac{\partial R}{\partial y} + \frac{\partial P}{\partial x} = f(x,y), \tag{2.9}$$

and $\delta$ stands for arbitrary independent variations

$$\delta x, \quad \delta y, \quad \delta u, \quad \delta v(a,b,\tau) \tag{2.10}$$

in the particle locations and velocities. These variations yield

$$\left.\begin{aligned} \delta x:&\quad \frac{\partial u}{\partial\tau} - f\frac{\partial y}{\partial\tau} = -g\frac{\partial h}{\partial x}, \\ \delta y:&\quad \frac{\partial v}{\partial\tau} + f\frac{\partial x}{\partial\tau} = -g\frac{\partial h}{\partial y}, \\ \delta u:&\quad u = \frac{\partial x}{\partial\tau}, \\ \delta v:&\quad v = \frac{\partial y}{\partial\tau}, \end{aligned}\right\} \tag{2.11}$$

which are equivalent to (2.2). The conservation of energy,

$$\mathrm{d}H/\mathrm{d}t = 0, \tag{2.12}$$

and the potential vorticity on particles,

$$\frac{\partial}{\partial \tau}\left[\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)\Big/h\right]=0, \tag{2.13}$$

correspond to the symmetry properties that $H$ is invariant to translations in time and to particle relabellings that do not affect the Jacobian (2.5). For complete details refer to S83.

In this paper, I am solely concerned with approximations to (2.7) that are valid in the limit of nearly geostrophic flow. If $u$ and $v$ are simply set equal to zero in (2.7), then the resulting Lagrangian

$$L_0 \equiv \iint \mathrm{d}a\,\mathrm{d}b\left[-R(x,y)\frac{\partial x}{\partial \tau}+P(x,y)\frac{\partial y}{\partial \tau}-\frac{g}{2}\frac{\partial(a,b)}{\partial(x,y)}\right] \tag{2.14}$$

depends only on the particle locations. The approximate dynamics

$$\delta \int L_0\,\mathrm{d}\tau = 0 \tag{2.15}$$

are equivalent to the equations for geostrophic balance, namely

$$\left.\begin{aligned} \delta x{:}\quad & -f\frac{\partial y}{\partial \tau}=-g\frac{\partial h}{\partial x},\\ \delta y{:}\quad & f\frac{\partial x}{\partial \tau}=-g\frac{\partial h}{\partial y}. \end{aligned}\right\} \tag{2.16}$$

Since mass conservation is implicit, (2.16) are equivalent to the following set of Eulerian equations:

$$\left.\begin{aligned} -fv &= -g\frac{\partial h}{\partial x},\\ fu &= -g\frac{\partial h}{\partial y},\\ \frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(uh)+\frac{\partial}{\partial y}(vh) &= 0. \end{aligned}\right\} \tag{2.17}$$

Equations (2.17) differ from (2.2) in the total neglect of the relative accelerations. This neglect is too severe for most applications in geophysical fluid dynamics. Suppose then that $u$ and $v$ are not dropped from (2.7), but are replaced *a priori* by their geostrophic values. The resulting Lagrangian

$$L_1 \equiv \iint \mathrm{d}a\,\mathrm{d}b\left[(u_{\mathrm{G}}-R)\frac{\partial x}{\partial \tau}+(v_{\mathrm{G}}+P)\frac{\partial y}{\partial \tau}-\frac{1}{2}\left(u_{\mathrm{G}}^2+v_{\mathrm{G}}^2+g\frac{\partial(a,b)}{\partial(x,y)}\right)\right] \tag{2.18}$$

still depends only on the particle locations, because the geostrophic velocities

$$u_{\mathrm{G}} \equiv \frac{-g}{f}\frac{\partial h}{\partial y}, \quad v_{\mathrm{G}} \equiv \frac{g}{f}\frac{\partial h}{\partial x} \tag{2.19}$$

are determined by the mass distribution. The integrand of $L_1$ differs from the exact integrand of $L$ by terms of order $\epsilon U^2$, where

$$\epsilon = U/f_0 L \tag{2.20}$$

is the Rossby number and $U, f_0$ and $L$ are the scales for velocity, Coriolis parameter and horizontal distance. As shown in S83, the approximate dynamics

$$\delta \int L_1 \,\mathrm{d}\tau = 0 \tag{2.21}$$

exactly conserves the geostrophy energy

$$H_1 \equiv \frac{1}{2}\iint \mathrm{d}a\,\mathrm{d}b\,[u_{\mathrm{G}}^2 + v_{\mathrm{G}}^2 + gh] \tag{2.22}$$

and a geostrophic approximation to the potential vorticity on fluid particles,

$$\frac{\partial}{\partial \tau}\left[\left(\frac{\partial v_{\mathrm{G}}}{\partial x} - \frac{\partial u_{\mathrm{G}}}{\partial y} + f\right)\Big/ h\right] = 0. \tag{2.23}$$

These laws are easily proved from the time and particle-label symmetries of $L_1$. In conventional Eulerian notation, the $L_1$ dynamics is

$$h\left[\frac{\partial}{\partial t}\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{u}_{\mathrm{G}}\cdot\nabla\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{u}_{\mathrm{G}}\cdot\nabla\boldsymbol{u}_{\mathrm{AG}} + \boldsymbol{u}_{\mathrm{AG}}\cdot\nabla\boldsymbol{u}_{\mathrm{G}}\right] + f\boldsymbol{k}\times h(\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{u}_{\mathrm{AG}}) + g\nabla(\tfrac{1}{2}h^2)$$
$$= -g\nabla\left[h^2\boldsymbol{k}\cdot\nabla\times\left(\frac{\boldsymbol{u}_{\mathrm{AG}}}{f}\right)\right] - g\nabla(\tfrac{1}{2}h^2)\left[\boldsymbol{u}_{\mathrm{AG}}\times\nabla\left(\frac{1}{f}\right)\right]\cdot\boldsymbol{k} \tag{2.24}$$

and
$$\frac{\partial h}{\partial t} + \nabla\cdot[(\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{u}_{\mathrm{AG}})\,h] = 0, \tag{2.25}$$

where $\boldsymbol{k}$ is the vertical unit vector and

$$\boldsymbol{u}_{\mathrm{AG}} \equiv \frac{\partial \boldsymbol{x}}{\partial \tau} - \boldsymbol{u}_{\mathrm{G}} \tag{2.26}$$

is the ageostrophic velocity. There is no explicit equation for the time evolution of $\boldsymbol{u}_{\mathrm{AG}}$, but an equation determining $\boldsymbol{u}_{\mathrm{AG}}$ from $h$ can be found by first forming a second equation for $\partial\boldsymbol{u}_{\mathrm{G}}/\partial t$ from (2.25) and (2.19), and then requiring this second equation to be consistent with (2.24). There results a pair of linear elliptic equations for $\boldsymbol{u}_{\mathrm{AG}}$. In the case of constant Coriolis parameter (for example), the equation determining $u_{\mathrm{AG}}$ is

$$\left[\frac{gh}{f}\nabla^2 - f + 2v_{\mathrm{G}}\frac{\partial}{\partial x} - 2u_{\mathrm{G}}\frac{\partial}{\partial y}\right]u_{\mathrm{AG}} = \boldsymbol{u}_{\mathrm{G}}\cdot\nabla v_{\mathrm{G}}, \tag{2.27}$$

which has a unique solution for $u_{\mathrm{AG}}$, subject to the boundary conditions $u_{\mathrm{AG}} = 0$ as $x, y \rightarrow \infty$. For details of the derivation of (2.24) refer to Appendix A.

The approximations $L \approx L_0$ and $L \approx L_1$ can be viewed as projections of the fluid state vector in the infinite-dimensional phase space spanned by $\{x, y, u, v\}$ onto the subspace spanned by $\{x, y\}$. In the case of $L_0$, the projected coordinates $\{u, v\}$ are simply set equal to zero. In the case of $L_1$, the coordinates $\{u, v\}$ are replaced by the values they would have if the motion were exactly geostrophic.

The $L_1$ dynamics has the same-order accuracy in the Rossby number as (say) the quasigeostrophic equations. However, unlike the quasigeostrophic equations, the $L_1$ dynamics allows order-one variations in the fluid depth and the Coriolis parameter. In addition, the $L_1$ dynamics exactly conserves proper analogues of the energy and potential vorticity. If $f = f_0 + \beta y$, where $\beta$ is a small constant, and the $L_1$ equations are linearized about a state of rest and constant depth $h_0$, then solutions proportional to

$$\exp[\mathrm{i}(kx+ly-\omega t)] \tag{2.28}$$

obey the same Rossby-wave dispersion relation

$$\omega = \frac{-\beta k}{k^2+l^2+\lambda^2}, \quad \lambda^2 \equiv \frac{f_0^2}{gh_0} \tag{2.29}$$

as the exact equations (2.2).

## 3. Generalized semigeostrophic equations

The modifications to $L_1$ dynamics that will be described below were motivated by an illuminating geometrical view of Hamiltonian mechanics which has been nicely summarized by Greene (1982). Briefly, every Hamiltonian system is defined by precisely two geometrical objects: the Poisson-bracket operator and the Hamiltonian function itself. In terms of these objects, the dynamical equations can be cast into a tensorial form which is covariant with respect to arbitrary transformations of the phase coordinates. The Poisson-bracket operator takes its simplest form when the chosen coordinates are canonical. Given any Poisson-bracket operator, there are infinitely many sets of canonical coordinates, inter-related by canonical transformations.

The foregoing facts suggest the following strategy for simplifying the $L_1$ dynamics: to seek canonical coordinates for the $L_1$ system, and, from among all possible sets of canonical coordinates, to choose that set in which the Hamiltonian takes its simplest form.

Now, if no further modifications to the $L_1$-dynamics were allowed, then the foregoing strategy would be hopelessly difficult to pursue. However, as noted above, the Lagrangian $L_1$ is already in error by terms of order $\epsilon U^2$ in its integrand. I am therefore free to modify the integrand of $L_1$ arbitrarily by terms of this same order. As will now be shown, this freedom makes it extremely easy to pursue the strategy outlined above.

First consider $L_0$. If the Coriolis parameter is constant (i.e. $f = 2\Omega$) then

$$L_0 = \Omega \iint \mathrm{d}a\,\mathrm{d}b \left[ -y\frac{\partial x}{\partial \tau} + x\frac{\partial y}{\partial \tau} - \frac{g}{2\Omega}\frac{\partial(a,b)}{\partial(x,y)} \right] \tag{3.1}$$

is already in canonical form. The conjugate variables are simply $x$ and $y$.† Suppose then that $f(x,y)$ is non-constant. Let

$$x_0(x,y), \quad y_0(x,y) \tag{3.2}$$

be any two functions of $(x,y)$ for which

$$\frac{\partial(x_0,y_0)}{\partial(x,y)} = \frac{f(x,y)}{f_0}. \tag{3.3}$$

† Note that

$$-y\frac{\partial x}{\partial \tau} + x\frac{\partial y}{\partial \tau} = 2x\frac{\partial y}{\partial \tau} - \frac{\partial(yx)}{\partial \tau},$$

and that the last term can be dropped, because the variations allowed by Hamilton's principle are zero at the endpoints in time.

Then the coordinates

$$x_0(a,b,\tau), \quad y_0(a,b,\tau) \tag{3.4}$$

are canonical. To see that this is true, define $R$ and $P$ by

$$R(x,y) = \tfrac{1}{2}f_0\left[y_0\frac{\partial x_0}{\partial x} - x_0\frac{\partial y_0}{\partial x}\right], \quad P(x,y) = \tfrac{1}{2}f_0\left[-y_0\frac{\partial x_0}{\partial y} + x_0\frac{\partial y_0}{\partial y}\right]. \tag{3.5}$$

It follows from (3.3) and (3.5) that $R$ and $P$ satisfy the required condition (2.9). Then direct substitution of (3.5) into (2.14) leads to

$$L_0 = \frac{f_0}{2}\iint \mathrm{d}a\,\mathrm{d}b\left[-y_0\frac{\partial x_0}{\partial \tau} + x_0\frac{\partial y_0}{\partial \tau} - g\frac{f}{f_0^2}\frac{\partial(a,b)}{\partial(x_0,y_0)}\right]. \tag{3.6}$$

It is therefore a simple matter to transform $L_0$ into canonical form. However, relatively little is gained, because (2.16) are already so simple.

Now consider $L_1$. I first seek a transformation from old coordinates

$$x(a,b,\tau), \quad y(a,b,\tau)$$

to new coordinates $\quad x_s(a,b,\tau), \quad y_s(a,b,\tau)$

for which

$$\iint \mathrm{d}a\,\mathrm{d}b\,\{[u_G - R(x,y)]\,\delta x + [v_G + P(x,y)]\,\delta y\}$$

$$= \iint \mathrm{d}a\,\mathrm{d}b\,\{-R(x_s,y_s)\,\delta x_s + P(x_s,y_s)\,\delta y_s\} + \delta S, \tag{3.7}$$

where $\delta\boldsymbol{x}$ is arbitrary, and $\delta\boldsymbol{x}_s$ is the image of $\delta\boldsymbol{x}$ under the sought-for transformation. In (3.7) and everywhere below, the prescribed functions $R(\ ,\ )$, $P(\ ,\ )$ and $f(\ ,\ )$ always have the same dependence on their arguments. The quantity $S$ is an arbitrary functional of $\boldsymbol{x}$ or $\boldsymbol{x}_s$ whose presence has no effect on the dynamics.† If a transformation satisfying (3.7) can be found then the transformed $L_1$ dynamics take the form

$$-f(x_s,y_s)\frac{\partial y_s}{\partial \tau} = -\frac{\delta H_1}{\delta x_s}, \quad f(x_s,y_s)\frac{\partial x_s}{\partial \tau} = -\frac{\delta H_1}{\delta y_s}, \tag{3.8}$$

which is almost canonical. If exactly canonical coordinates are desired then $(x_s,y_s)$ may be subjected to a further transform like that from $(x,y)$ to $(x_0,y_0)$ above. This further transformation, which has the effect of replacing $f(x_s,y_s)$ in (3.8) by a constant, is probably not worth the extra trouble.

It is very difficult to find a coordinate transformation that satisfies (3.7) exactly. However, it is unnecessary to satisfy (3.7) exactly, because the integrand of $L_1$ already contains errors of order $\epsilon U^2$. It is therefore only necessary to satisfy the transformation condition (3.7) to within order $\epsilon UL$. This turns out to be very easy. Let

$$x_s = x + F, \quad y_s = y + G. \tag{3.9}$$

where $F$ and $G$ are functionals of order $\epsilon L$, to be determined. By direct substitution,

† This is because the variations allowed by Hamilton's principle are zero at the endpoints in time. In conventional theory, $S$ would be called the generating function of the transform. However, this terminology is inappropriate here, because the transformation is between non-canonical coordinates.

$$\iint \mathrm{d}a\,\mathrm{d}b\,[-R(x_s, y_s)\,\delta x_s + P(x_s, y_s)\,\delta y_s]$$

$$= \iint \mathrm{d}a\,\mathrm{d}b\left\{\left[-R(x,y) - \frac{\partial R}{\partial x}F - \frac{\partial R}{\partial y}G\right]\delta x\right.$$

$$\left. + \left[P(x,y) + \frac{\partial P}{\partial x}F + \frac{\partial P}{\partial y}G\right]\delta y - R(x,y)\,\delta F + P(x,y)\,\delta G\right\} + O(\epsilon^2 f_0 L^2)$$

$$= \iint \mathrm{d}a\,\mathrm{d}b\,\{[-R(x,y) - Gf(x,y)]\,\delta x + [P(x,y) + Ff(x,y)]\,\delta y$$

$$-\delta(RF) + \delta(PG)\} + O(\epsilon^2 f_0 L^2). \tag{3.10}$$

Here $O(\epsilon^2 f_0 L^2)$ stands for quadratic (and higher) terms in $F$ and $G$. The last two terms in (3.10) can be absorbed into the arbitrary functional $S$. The remaining terms match the left-hand side of (3.7) if

$$F = \frac{v_G}{f} + O\left(\epsilon\frac{U}{f_0}\right) \quad , \quad G = \frac{-u_G}{f} + O\left(\epsilon\frac{U}{f_0}\right). \tag{3.11}$$

In particular, the coordinates

$$x_s = x + \frac{v_s}{f(x_s, y_s)}, \quad y_s = y - \frac{u_s}{f(x_s, y_s)} \tag{3.12}$$

satisfy (3.11) to within the required accuracy. Here

$$u_s \equiv -\frac{g}{f(x_s, y_s)}\frac{\partial h}{\partial y}, \quad v_s \equiv \frac{g}{f(x_s, y_s)}\frac{\partial h}{\partial x}. \tag{3.13}$$

The choice (3.12) was made with the faith that the final transformed equations would take a simple form if the arguments of $R$, $P$ and $f$ were made the same in every term of the approximate Lagrangian. This turns out to be the case. Applying this principle also to the Hamiltonian $H_1$, I replace $H_1$ by

$$H_s \equiv \frac{1}{2}\iint \mathrm{d}a\,\mathrm{d}b\,[u_s^2 + v_s^2 + gh]. \tag{3.14}$$

The integrands of $H_s$ and $H_1$ also differ by terms of order $\epsilon U^2$. The final transformed dynamics are now

$$\delta\int L_s\,\mathrm{d}\tau = 0, \tag{3.15}$$

where
$$L_s = \iint \mathrm{d}a\,\mathrm{d}b\left[-R(x_s, y_s)\frac{\partial x_s}{\partial \tau} + P(x_s, y_s)\frac{\partial y_s}{\partial \tau}\right] - H_s. \tag{3.16}$$

The variations $\delta x_s(a, b, \tau)$, $\delta y_s(a, b, \tau)$ yield

$$\left.\begin{aligned} \delta x_s\!: \quad & -f(x_s, y_s)\frac{\partial y_s}{\partial \tau} = -\frac{\delta H_s}{\delta x_s}, \\ \delta y_s\!: \quad & f(x_s, y_s)\frac{\partial x_s}{\partial \tau} = -\frac{\delta H_s}{\delta y_s}. \end{aligned}\right\} \tag{3.17}$$

The dynamics (3.17) has the same accuracy as the $L_1$ dynamics. This is true because the integrands of $L_s$ and $L_1$ differ by order $\epsilon U^2$, the same size difference as between

$L_1$ and the exact Lagrangian $L$. Moreover, since the time and particle-label symmetries have not been disturbed, the dynamics (3.17) exactly conserves the energy $H_s$ and the following form of potential vorticity on fluid particles:

$$\begin{aligned}\frac{f(x_s,y_s)}{h_s} &= \frac{f(x_s,y_s)}{h}\frac{\partial(x_s,y_s)}{\partial(x,y)} \\ &= \frac{1}{h}\left[f(x,y)+\frac{\partial f}{\partial x}\frac{v_s}{f_s}-\frac{\partial f}{\partial y}\frac{u_s}{f_s}+O(\epsilon^2 f_0)\right]\left[1+\frac{\partial}{\partial x}\left(\frac{v_s}{f_s}\right)-\frac{\partial}{\partial y}\left(\frac{u_s}{f_s}\right)+O(\epsilon^2)\right] \\ &= \frac{f(x,y)+\dfrac{\partial v_G}{\partial x}-\dfrac{\partial u_G}{\partial y}}{h}[1+O(\epsilon^2)],\end{aligned} \tag{3.18}$$

where
$$h_s \equiv \frac{\partial(a,b)}{\partial(x_s,y_s)}, \quad f_s \equiv f(x_s,y_s). \tag{3.19}$$

Thus (3.18) is a consistent low-Rossby-number approximation to the exact potential vorticity in (2.13).

The functional derivatives in (3.17) can be evaluated, and they take a simple form (see Appendix B). It turns out that

$$\frac{\delta H_s}{\delta x_s} = \frac{\partial \Phi_s}{\partial x_s}, \quad \frac{\delta H_s}{\delta y_s} = \frac{\partial \Phi_s}{\partial y_s}, \tag{3.20}$$

where
$$\Phi_s \equiv \tfrac{1}{2}(u_s^2+v_s^2)+gh. \tag{3.21}$$

The final $L_s$ dynamics thus take the form

$$-f_s\frac{\partial y_s}{\partial \tau} = -\frac{\partial \Phi_s}{\partial x_s}, \quad f_s\frac{\partial x_s}{\partial \tau} = -\frac{\partial \Phi_s}{\partial y_s}. \tag{3.22}$$

In the case of a constant Coriolis parameter, (3.12), (3.21), and (3.22) reduce to

$$x_s = x+\frac{v_G}{f_0}, \quad y_s = y-\frac{u_G}{f_0}, \tag{3.23}$$

$$-f_0\frac{\partial y_s}{\partial \tau} = -\frac{\partial \Phi_s}{\partial x_s} = -f_0 v_G, \quad f_0\frac{\partial x_s}{\partial \tau} = -\frac{\partial \Phi_s}{\partial y_s} = f_0 u_G, \tag{3.24}$$

and
$$\Phi_s = \tfrac{1}{2}(u_G^2+v_G^2)+gh. \tag{3.25}$$

The final equalities in (3.24) are proved in Appendix B. The equations (3.23)–(3.25) are precisely equivalent to the semigeostrophic equations of Hoskins (1975). The semigeostrophic equations exactly conserve the energy $H_1$ and the following form of the potential vorticity on fluid particles:

$$\frac{f_0}{h_s} = \left[f_0+\frac{\partial v_G}{\partial x}-\frac{\partial u_G}{\partial y}+\frac{1}{f_0}\frac{\partial(u_G,v_G)}{\partial(x,y)}\right]\Big/ h. \tag{3.26}$$

Again, (3.26) is a consistent low-Rossby-number approximation to the exact potential vorticity in (2.13).

The generalized semigeostrophic equations (3.22) can be solved as follows. Let $\mathbf{x}_s(a,b,\tau_0)$ and $\Phi_s(x_s,y_s,\tau_0)$ be given at the initial time $\tau_0$. Use (3.22) to obtain $\mathbf{x}_s(a,b,\tau_0+\Delta\tau)$ at the new time $\tau_0+\Delta\tau$. This process can be continued only if

$\Phi_s(x_s, y_s, \tau_0+\Delta\tau)$ can be found. To determine $\Phi_s$, solve the transformation equations (3.12) for the untransformed particle locations $\boldsymbol{x}(a, b, \tau_0+\Delta\tau)$. Then $h$, its $x$- and $y$-derivatives, and hence $\Phi_s$ can be computed.

Hoskins has suggested a specific method for determining $\Phi_s$ which is interesting for two reasons. First, it shows that the semigeostrophic equations can be closed in the transformed variables. Secondly, it demonstrates an interesting connection with the ordinary quasigeostrophic equations. As noted by Hoskins, the conservation of potential vorticity

$$q \equiv f_s/h_s \tag{3.27}$$

may be expressed as

$$\left[\frac{\partial}{\partial t_s}+\frac{\partial x_s}{\partial \tau}\frac{\partial}{\partial x_s}+\frac{\partial y_s}{\partial \tau}\frac{\partial}{\partial y_s}\right]q = 0. \tag{3.28}$$

Here $t_s = \tau$, but $(x_s, y_s, t_s)$ are independent variables. By (3.22), (3.28) becomes

$$\frac{\partial q}{\partial t_s}+\frac{1}{f_s}\frac{\partial(\Phi_s, q)}{\partial(x_s, y_s)} = 0, \tag{3.29}$$

which can be used to step $q(x_s, y_s, t_s)$ forward in time. Then the problem is to determine $\Phi_s(x_s, y_s, t_s)$ from $q$ at the new time. Now

$$\frac{gf_s}{q} = gh_s = gh\frac{\partial(x, y)}{\partial(x_s, y_s)} \tag{3.30}$$

is equivalent to

$$\frac{gf_s}{q} = [\Phi_s-\tfrac{1}{2}(u_s^2+v_s^2)]\left[1-\frac{\partial}{\partial x_s}\left(\frac{v_s}{f_s}\right)+\frac{\partial}{\partial y_s}\left(\frac{u_s}{f_s}\right)+\frac{\partial(u_s/f_s, v_s/f_s)}{\partial(x_s, y_s)}\right] \tag{3.31}$$

after substitutions from (3.21) and (3.12). But, as shown in Appendix B,

$$\nabla_s\Phi_s = f_s(v_s, -u_s)+\frac{u_s^2+v_s^2}{f_s}\nabla_s f_s. \tag{3.32}$$

Elimination of $u_s$, $v_s$ between (3.32) and (3.31) gives a nonlinear elliptic equation which determines $\Phi_s$ from $q$. These equations can be solved by iterations, because the nonlinear terms in (3.31) and (3.32) are of higher order in the Rossby number. To a first approximation in the Rossby number, (3.31) and (3.32) reduce to

$$f_s\left[1+\frac{\partial}{\partial x_s}\left(\frac{1}{f_s^2}\frac{\partial\Phi_s}{\partial x_s}\right)+\frac{\partial}{\partial y_s}\left(\frac{1}{f_s^2}\frac{\partial\Phi_s}{\partial y_s}\right)\right] = \frac{q\Phi_s}{g}. \tag{3.33}$$

Now if, as assumed in the quasigeostrophic approximation, the lengthscale for variation of the Coriolis parameter is very large, and the departure $\Phi_s'$ of $\Phi_s$ from its constant mean value $\Phi_s^0$ is small, then a consistent low-Rossby-number approximation to (3.33) is

$$\left[f_s+\frac{1}{f_0}\nabla_s^2\Phi_s'-\frac{f_0}{\Phi_s^0}\Phi_s'\right] = \frac{\Phi_s^0}{g}q. \tag{3.34}$$

Equations (3.29) and (3.34) are formally identical with the quasigeostrophic equation, except that $\Phi_s'$ replaces the ordinary stream function and $(x_s, y_s, t_s)$ replace the ordinary variables $(x, y, t)$. Solutions of the generalized semigeostrophic equations therefore resemble solutions of the quasigeostrophic equation, except that, as noted by Hoskins, the transformation (3.12) to physical space causes a distortion in which regions of positive relative vorticity become smaller and regions of negative relative vorticity become larger. This asymmetry between low- and high-pressure centres is, of course, a characteristic property of weather maps.

In the case of constant Coriolis parameter, the semigeostrophic equations (3.24) are precisely equivalent to the Eulerian equations

$$\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{u}_{\mathrm{G}}+f\boldsymbol{k}\times\boldsymbol{u}=-g\,\nabla h. \tag{3.35}$$

The only difference between (3.35) and the exact equations (2.2) is that the geostrophic velocity $\boldsymbol{u}_{\mathrm{G}}$ replaces the exact velocity $\boldsymbol{u}$ after the substantial derivative. This has been called the 'geostrophic momentum approximation'. In the general case of a non-constant Coriolis parameter, the semigeostrophic equations (3.22) take the same Eulerian form (3.35) *within error terms* of order $\epsilon^2 f_0 U$. This can be proved by direct substitutions from (3.12), (3.13) and (3.21), and numerous algebraic cancellations.

## 4. New equations for large-scale flow

The equations (2.17) for purely geostrophic motion have been used to study flows with dominant lengthscales greater than the deformation radius $r$, where

$$r\equiv\lambda^{-1}=(gh_0)^{\frac{1}{2}}/f_0. \tag{4.1}$$

For example, (2.17) apply to an ocean composed of two immiscible layers in which the lower layer is everywhere at rest. In this application, $(u,v)$ and $h$ represent the velocity and depth of the upper layer and $g$ is replaced by the reduced gravity $g'$, where

$$g'=g\,\Delta\rho/\rho \tag{4.2}$$

and $\Delta\rho$ is the small density difference between layers. The corresponding deformation radius is about 40 km. This model (with appropriate wind-forcing and friction terms appended) and its multilayer generalizations have frequently been used to study the large-scale mean ocean circulation (see e.g. Parsons 1969). If the term $\partial h/\partial t$ is struck from (2.17*c*), then (2.17) are the simplest case of the 'thermocline equations' (Pedlosky 1979).

Even in the case of very large-scale flows, it may be incorrect to neglect the relative accelerations completely. For example, it is widely thought that the ocean boundary layers have an inertial character of the type first considered by Fofonoff (1954). The interior flow may be accurately governed by (2.17) but this flow is greatly affected by the presence of the boundary layers. In this context, the term 'boundary layer' also applies to narrow intense currents like the Gulf Stream after they have detached from the coast. The Fofonoff boundary-layer thickness is unrelated to the deformation radius.

In this section I derive new equations for large-scale flow which are hardly more complicated (in transformed coordinates) than (2.17), but consistently include the effects of relative accelerations on the large-scale flow. First, define

$$u_{\mathrm{s}}^*\equiv\frac{-g}{f_{\mathrm{s}}}\frac{\partial h_{\mathrm{s}}}{\partial y_{\mathrm{s}}},\quad v_{\mathrm{s}}^*\equiv\frac{g}{f_{\mathrm{s}}}\frac{\partial h_{\mathrm{s}}}{\partial x_{\mathrm{s}}}, \tag{4.3}$$

and note that $(u_{\mathrm{s}}^*, v_{\mathrm{s}}^*)$ differ from $(u_{\mathrm{s}}, v_{\mathrm{s}})$ in that $(x_{\mathrm{s}}, y_{\mathrm{s}}, h_{\mathrm{s}})$ replace $(x,y,h)$. It is easy to show that

$$h=h_{\mathrm{s}}\left[1+\frac{\partial}{\partial x}\left(\frac{v_{\mathrm{s}}}{f_{\mathrm{s}}}\right)-\frac{\partial}{\partial y}\left(\frac{u_{\mathrm{s}}}{f_{\mathrm{s}}}\right)+O(\epsilon^2)\right], \tag{4.4}$$

so that $$u_s = u_s^* + O(\epsilon U) + O(BU), \quad v_s = v_s^* + O(\epsilon U) + O(BU), \tag{4.5}$$

and $$gh = gh_s\left[1 + \frac{\partial}{\partial x_s}\left(\frac{v_s^*}{f_s}\right) - \frac{\partial}{\partial y_s}\left(\frac{u_s^*}{f_s}\right)\right] + O(BU^2) + O\left(\frac{B^2U^2}{\epsilon}\right), \tag{4.6}$$

where $$B \equiv \frac{gh_0}{f_0^2 L^2} = \frac{r^2}{L^2} \tag{4.7}$$

is the 'Burger' number. It follows from (4.5) and (4.6) that

$$\begin{aligned} H_s^* &\equiv \frac{1}{2}\iint \mathrm{d}a\,\mathrm{d}b\left\{u_s^{*2} + v_s^{*2} + gh_s\left[1 + \frac{\partial}{\partial x_s}\left(\frac{v_s^*}{f_s}\right) - \frac{\partial}{\partial y_s}\left(\frac{u_s^*}{f_s}\right)\right]\right\} \\ &= \frac{1}{2}\iint \mathrm{d}a\,\mathrm{d}b\,[gh_s - u_s^{*2} - v_s^{*2}] \end{aligned} \tag{4.8}$$

is a consistent approximation to $H_s$ for $B = O(\epsilon)$, i.e. small Rossby number *and* small Burger number. The last equality in (4.8) follows an integration by parts. The approximate dynamics

$$\delta \int L_s^*\,\mathrm{d}\tau = 0, \tag{4.9}$$

where $$L_s^* = \iint \mathrm{d}a\,\mathrm{d}b\left[-R(x_s, y_s)\frac{\partial x_s}{\partial \tau} + P(x_s, y_s)\frac{\partial y_s}{\partial \tau}\right] - H_s^*, \tag{4.10}$$

has the same accuracy as $L_1$ dynamics and the generalized semigeostrophic dynamics at lengthscales larger than the deformation radius. Note that $H_s^*$, unlike $H_s$, has a simple dependence on the transformed particle locations $\boldsymbol{x}_s(a, b, \tau)$. As shown below, this leads to simple closed equations in the transformed variables. The variational equations corresponding to (4.9) are

$$-f_s\frac{\partial y_s}{\partial \tau} = -\frac{\delta H_s^*}{\delta x_s}, \quad f_s\frac{\partial x_s}{\partial \tau} = -\frac{\delta H_s^*}{\delta y_s}. \tag{4.11}$$

As shown in Appendix C, the functional derivatives in (4.11) again take the form

$$\frac{\delta H_s^*}{\delta x_s} = \frac{\partial \Phi_s^*}{\partial x_s}, \quad \frac{\delta H_s^*}{\delta y_s} = \frac{\partial \Phi_s^*}{\partial y_s}, \tag{4.12}$$

where now $$\Phi_s^* \equiv \tfrac{1}{2}u_s^{*2} + \tfrac{1}{2}v_s^{*2} + gh_s\left[1 + \frac{\partial}{\partial x_s}\left(\frac{v_s^*}{f_s}\right) - \frac{\partial}{\partial y_s}\left(\frac{u_s^*}{f_s}\right)\right]. \tag{4.13}$$

From (4.3) and (4.13) it follows that the potential $\Phi_s^*$ depends *only* on $h_s$ and its derivatives in the transformed variables $(x_s, y_s)$. Because of this fact, it is possible to cast the $L_s^*$ dynamics into the form of a single *prognostic* equation for $h_s$. There is no elliptic equation to solve and no need to solve the transformation equations until it is time 'to look at the answer'. To appreciate these facts, first note that a direct application of $\partial/\partial\tau$ to the definition

$$h_s \equiv \frac{\partial(a, b)}{\partial(x_s, y_s)}$$

yields an exact equation for the conservation of mass:

$$\frac{\partial h_s}{\partial t_s} + \frac{\partial}{\partial x_s}\left(h_s\frac{\partial x_s}{\partial \tau}\right) + \frac{\partial}{\partial y_s}\left(h_s\frac{\partial y_s}{\partial \tau}\right) = 0. \tag{4.14}$$

Substitutions from (4.11) and (4.12) bring (4.14) into the form

$$\frac{\partial h_s}{\partial t_s}+\frac{\partial(\Phi_s^*, h_s/f_s)}{\partial(x_s, y_s)}=0, \tag{4.15}$$

which contains only $h_s(x_s, y_s, t_s)$ and its derivatives. The potential-vorticity equation

$$\left[\frac{\partial}{\partial t_s}+\frac{\partial x_s}{\partial \tau}\frac{\partial}{\partial x_s}+\frac{\partial y_s}{\partial \tau}\frac{\partial}{\partial y_s}\right]\frac{f_s}{h_s}=0 \tag{4.16}$$

could also be used, instead of (4.14).

Suppose $B = O(\epsilon)$. Then the dynamics (4.11), (4.12) or (4.15) exactly conserves the total energy

$$H_s^* = \frac{1}{2}\iint \mathrm{d}a\,\mathrm{d}b\,[u_G^2+v_G^2+gh+O(\epsilon U^2)] \tag{4.17}$$

and the following form of the potential vorticity on particles:

$$\frac{f(x_s, y_s)}{h_s}=\frac{\left[f(x,y)+\dfrac{\partial v_G}{\partial x}-\dfrac{\partial u_G}{\partial y}\right]}{h}[1+O(\epsilon^2)]. \tag{4.18}$$

These conservation laws are easily proved from the time and particle-label symmetries of $L_s^*$ or directly from the dynamics (4.11)–(4.15). Direct substitutions verify that these dynamics are equivalent to

$$\frac{\partial}{\partial t}\boldsymbol{u}_G+\boldsymbol{u}_G\cdot\nabla\boldsymbol{u}_G+f\boldsymbol{k}\times\boldsymbol{u}=-g\nabla h[1+O(\epsilon^2)] \tag{4.19}$$

in conventional Eulerian notation. Here $O(\epsilon^2)$ stands for higher-order terms, which are of the order of the error in the approximation $L \approx L_s^*$, but which must be included for the exact conservation laws to obtain. The transformed equations (4.11), (4.12) or (4.15) are much the simplest way to pose $L_s^*$ dynamics.

If (4.15) is linearized about a state of rest and constant depth, then the linear wave solutions of (4.15) obey the dispersion relation

$$\omega=\frac{-\beta k}{\lambda^2}\left[1-\frac{(k^2+l^2)}{\lambda^2}\right], \tag{4.20}$$

which is a consistent approximation to (2.29) for large-scale waves ($k^2+l^2 < \lambda^2$). For vanishing wavelengths, to which the $L_s^*$ dynamics do not accurately apply, the phase and group velocities corresponding to (4.20) become infinite. This explains how it has been possible to include the effects of the relative vorticity without solving an elliptic equation like (2.27) or (3.34). The $L_s^*$ dynamics are appropriate for basin-scale numerical models of the ocean. In even the largest of these models, the deformation radius is barely resolved.

## 5. Final comments

In the commonest procedure for obtaining approximate dynamical equations, one begins with the 'exact' equations of motion in some particular (usually Eulerian) coordinate system. A scaling analysis identifies some of the terms in these equations as 'small'. The small terms are then neglected on the tacit assumption that small errors in the equations of motion cause only small errors in the solutions to these equations. This assumption is, however, generally untrue. It is well known, for example, that even very small errors in the initial conditions of a turbulent flow cause order-one errors in the flow after a finite time. The neglect of small terms in the equations of motion is obviously equivalent to a continuously acting source of small errors. Therefore, the neglect of small terms cannot *generally* yield solutions that are close to the exact solutions, except in some imprecisely defined average sense.

The equations governing dynamical systems always have an underlying Hamiltonian structure and an associated system of symmetry properties and conservation laws. Recent research on dynamical systems has only reemphasized the importance of these characteristics in determining the behaviour of the dynamical system. I suggest that dynamical approximations should always preserve this Hamiltonian structure and retain analogues of all the exact conservation laws. The combination of formal accuracy *plus* the proper conservation laws is a better guarantee of an acceptable approximation than is formal accuracy by itself.

Lorenz (1960) was among the first to realize that approximations based solely upon a scaling analysis do not generally maintain analogues of the exact conservation laws. Lorenz showed that small terms in the Eulerian fluid equations must be omitted or retained in special combinations, or the conservation laws are lost. In the Hamiltonian methods of this paper, the conservation laws are automatically maintained because approximations based upon a scaling analysis are applied directly to the Lagrangian, taking care not to break the symmetry properties corresponding to the conservation laws of the fluid. My results are a generalization of the results of Lorenz and others, in the sense that they allow the *appearance* as well as the disappearance of small error terms in the Eulerian equations and in the expressions for conserved quantities like potential vorticity.

Every dynamical approximation has two distinct elements: the inherent physics of the approximation, and the coordinates used to describe it. The *accuracy* of the approximation, and the *existence* of conservation laws, are covariant properties of the *physics*: they are not affected by transformations to new coordinates. On the other hand, the *mathematical simplicity* of the approximation is highly dependent on the choice of coordinates, and can only be judged in that particular set of coordinates in which the chosen physics takes its simplest form. There is no reason to favour any other set of coordinates. In this paper, I have shown how an opportunistic, bootstrapping approach, based on an appreciation for the Hamiltonian structure of fluid mechanics, in which the physics and coordinates are simultaneously adjusted, can lead to physically consistent approximations of surprising simplicity.

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## Appendix A

Let

$$A = \int L_1, \tag{A 1}$$

where

$$\int \equiv \int\int\int \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}\tau \tag{A 2}$$

and $L_1$ is given by (2.18). For arbitrary variations $\delta \boldsymbol{x}(a, b, \tau)$,

$$\delta A = \int \left\{ (u_G - R)\frac{\partial}{\partial \tau}\delta x + (v_G + P)\frac{\partial}{\partial \tau}\delta y - \dot{x}\,\delta R + \dot{y}\,\delta P + (\dot{\boldsymbol{x}} - \boldsymbol{u}_G)\cdot\delta\boldsymbol{u}_G - \tfrac{1}{2}g\,\delta h \right\} \tag{A 3}$$

$$= \int \{ -(\dot{u}_G - \dot{R})\,\delta x - (\dot{v}_G + \dot{P})\,\delta y - \dot{x}(\boldsymbol{\nabla} R\cdot\delta\boldsymbol{x}) + \dot{y}(\boldsymbol{\nabla} P\cdot\delta\boldsymbol{x}) + \boldsymbol{u}_{AG}\cdot\delta\boldsymbol{u}_G - \tfrac{1}{2}g\,\delta h \}. \tag{A 4}$$

Here
$$(\dot{\ }) \equiv \frac{\partial}{\partial \tau}, \tag{A 5}$$

and $h$ and $\boldsymbol{u}_{AG}$ are defined by (2.5) and (2.26). Using

$$\dot{R} = \boldsymbol{\nabla} R\cdot\dot{\boldsymbol{x}}, \quad \dot{P} = \boldsymbol{\nabla} P\cdot\dot{\boldsymbol{x}} \tag{A 6}$$

and (2.9), (A 4) may be simplified to

$$\delta A = \int \{ (-\dot{u}_G + f\dot{y})\,\delta x + (-\dot{v}_G - f\dot{x})\,\delta y + \boldsymbol{u}_{AG}\cdot\delta\boldsymbol{u}_G - \tfrac{1}{2}g\,\delta h \}. \tag{A 7}$$

Now for any scalar $F$,

$$\begin{aligned}
\int F\,\delta h &= -\int F h^2\,\delta\frac{\partial(x,y)}{\partial(a,b)} \\
&= -\int F h^2 \left[\frac{\partial(\delta x, y)}{\partial(a,b)} + \frac{\partial(x,\delta y)}{\partial(a,b)}\right] \\
&= +\int \left[\frac{\partial(Fh^2, y)}{\partial(a,b)}\,\delta x + \frac{\partial(x, Fh^2)}{\partial(a,b)}\,\delta y\right] \\
&= \int \frac{1}{h}\,\boldsymbol{\nabla}(Fh^2)\cdot\delta\boldsymbol{x}.
\end{aligned} \tag{A 8}$$

A similar (but much lengthier) calculation establishes that

$$\int \boldsymbol{F}\cdot\delta\boldsymbol{u}_G = \int \mathrm{d}\boldsymbol{x}\cdot\left\{ -(\boldsymbol{u}_G\cdot\boldsymbol{\nabla})\boldsymbol{F} - g\left[\boldsymbol{k}\cdot\left(\boldsymbol{F}\times\boldsymbol{\nabla}\left(\frac{1}{f}\right)\right)\right]\boldsymbol{\nabla} h - \frac{g}{h}\boldsymbol{\nabla}\left[h^2\boldsymbol{k}\cdot\boldsymbol{\nabla}\times\frac{\boldsymbol{F}}{f}\right]\right\} \tag{A 9}$$

for any vector $\boldsymbol{F}$. Setting $F = -\tfrac{1}{2}g$ in (A 8) and $\boldsymbol{F} = \boldsymbol{u}_{AG}$ in (A 9), substituting the results into (A 7), and equating coefficients of $\delta\boldsymbol{x}$ to zero, I finally obtain (2.24).

## Appendix B

The existence of a function $\Phi_s$ satisfying (3.20) can be anticipated as follows. Direct application of $\partial/\partial\tau$ to the definition

$$h_s \equiv \frac{\partial(a,b)}{\partial(x_s, y_s)} \tag{B 1}$$

yields an exact equation for the conservation of mass in transformed coordinates, namely

$$\frac{\partial h_s}{\partial t_s} + \frac{\partial}{\partial x_s}\left(h_s\frac{\partial x_s}{\partial \tau}\right) + \frac{\partial}{\partial y_s}\left(h_s\frac{\partial y_s}{\partial \tau}\right) = 0, \tag{B 2}$$

where $t_s = \tau$ but $(x_s, y_s, t_s)$ are independent variables. On the other hand, the conservation of potential vorticity $f_s/h_s$ can be expressed as

$$\left[\frac{\partial}{\partial t_s}+\frac{\partial x_s}{\partial \tau}\frac{\partial}{\partial x_s}+\frac{\partial y_s}{\partial \tau}\frac{\partial}{\partial y_s}\right]\frac{f_s}{h_s}=0. \tag{B 3}$$

The two equations (B 2) and (B 3) are compatible only if

$$-f_s\frac{\partial y_s}{\partial \tau}=-\frac{\partial \Phi_s}{\partial x_s}, \quad f_s\frac{\partial x_s}{\partial \tau}=-\frac{\partial \Phi_s}{\partial y_s} \tag{B 4}$$

for some $\Phi_s$.

To verify (3.20) and (3.21) directly, first note that

$$\begin{aligned}\delta H_s &= \iint \mathrm{d}a\,\mathrm{d}b\,(u_s\,\delta u_s+v_s\,\delta v_s+\tfrac{1}{2}g\,\delta h] \\ &= \iint \mathrm{d}a\,\mathrm{d}b\left[u_s\,\delta u_s+v_s\,\delta v_s+g\frac{\partial h}{\partial x}\,\delta x+g\frac{\partial h}{\partial y}\,\delta y\right],\end{aligned} \tag{B 5}$$

where the last equality follows an integration by parts. By use of the transformation equations (3.12), this becomes

$$\begin{aligned}\delta H_s &= \iint \mathrm{d}a\,\mathrm{d}b\left[u_s\,\delta u_s+v_s\,\delta v_s+f_s\,v_s\,\delta\left(x_s-\frac{v_s}{f_s}\right)-f_s\,u_s\,\delta\left(y_s+\frac{u_s}{f_s}\right)\right] \\ &= \iint \mathrm{d}a\,\mathrm{d}b\left[v_s f_s\,\delta x_s-u_s f_s\,\delta y_s+(u_s^2+v_s^2)\frac{\delta f_s}{f_s}\right].\end{aligned} \tag{B 6}$$

Therefore
$$\frac{\delta H_s}{\delta x_s}=v_s f_s+(u_s^2+v_s^2)f_s^{-1}\frac{\partial f_s}{\partial x_s}, \tag{B 7}$$

and similarly for $\delta H_s/\delta y_s$. On the other hand,

$$\begin{aligned}\frac{\partial \Phi_s}{\partial x_s} &= \frac{\partial}{\partial x_s}(\tfrac{1}{2}u_s^2+\tfrac{1}{2}v_s^2+gh) \\ &= u_s\frac{\partial u_s}{\partial x_s}+v_s\frac{\partial v_s}{\partial x_s}+g\frac{\partial h}{\partial x_s}.\end{aligned} \tag{B 8}$$

However,

$$\begin{aligned}g\frac{\partial h}{\partial x_s} &= g\frac{\partial h}{\partial x}\frac{\partial x}{\partial x_s}+g\frac{\partial h}{\partial y}\frac{\partial y}{\partial x_s} \\ &= f_s\,v_s\frac{\partial}{\partial x_s}\left(x_s-\frac{v_s}{f_s}\right)-f_s\,u_s\frac{\partial}{\partial x_s}\left(y_s+\frac{u_s}{f_s}\right).\end{aligned} \tag{B 9}$$

Substitution of (B 9) into (B 8) and comparison with (B 7) establishes that

$$\frac{\delta H_s}{\delta x_s}=\frac{\partial \Phi_s}{\partial x_s}. \tag{B 10}$$

## Appendix C

Let

$$H = \iint \mathrm{d}a\,\mathrm{d}b\,\phi(u,v,h), \tag{C 1}$$

where $\phi$ is an arbitrary function of the variables

$$u = -\frac{g}{f}\frac{\partial h}{\partial y}, \quad v = \frac{g}{f}\frac{\partial h}{\partial x}, \quad h = \frac{\partial(a,b)}{\partial(x,y)}. \tag{C 2}$$

Consider variations in the mapping from $(a,b)$ to $(x,y)$. It is easiest to regard $(x,y)$ as fixed. Then

$$\begin{aligned}
\delta H &= \delta \iint \mathrm{d}x\,\mathrm{d}y\,h\phi \\
&= \iint \mathrm{d}x\,\mathrm{d}y\,[\delta h\cdot\phi + h(\phi_h\,\delta h + \phi_u\,\delta u + \phi_v\,\delta v)] \\
&= \iint \mathrm{d}x\,\mathrm{d}y\left\{[\phi+\phi_h h]\,\delta h + \frac{gh}{f}\left[\phi_v\frac{\partial\delta h}{\partial x} - \phi_u\frac{\partial \delta h}{\partial y}\right]\right\} \\
&= \iint \mathrm{d}x\,\mathrm{d}y\,\delta h\,\Phi \\
&= \iint \mathrm{d}a\,\mathrm{d}b\left[\frac{\partial \delta a}{\partial a} + \frac{\partial \delta b}{\partial b}\right]\Phi \\
&= -\iint \mathrm{d}a\,\mathrm{d}b\left[\delta a\frac{\partial}{\partial a} + \delta b\frac{\partial}{\partial b}\right]\Phi,
\end{aligned} \tag{C 3}$$

where

$$\Phi \equiv \phi + \phi_h h + \frac{\partial}{\partial y}\left(\frac{gh}{f}\phi_u\right) - \frac{\partial}{\partial x}\left(\frac{gh}{f}\phi_v\right). \tag{C 4}$$

But

$$\delta a = -\frac{\partial a}{\partial x}\delta x - \frac{\partial a}{\partial y}\delta y, \quad \delta b = -\frac{\partial b}{\partial x}\delta x - \frac{\partial b}{\partial y}\delta y, \tag{C 5}$$

and thus

$$\delta H = \iint \mathrm{d}a\,\mathrm{d}b\left[\delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y}\right]\Phi. \tag{C 6}$$

Therefore

$$\frac{\delta H}{\delta x} = \frac{\partial \Phi}{\partial x}, \quad \frac{\delta H}{\delta y} = \frac{\partial \Phi}{\partial y}. \tag{C 7}$$

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