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Weakly dispersive nonlinear gravity waves

By JOHN MILES AND RICK SALMON
Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, California 92093

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The equations for gravity waves on the free surface of a laterally unbounded inviscid fluid of uniform density and variable depth under the action of an external pressure are derived through Hamilton's principle on the assumption that the fluid moves in vertical columns. The resulting equations are equivalent to those of Green & Naghdi (1976). The conservation laws for energy, momentum and potential vorticity are inferred directly from symmetries of the Lagrangian. The potential vorticity vanishes in any flow that originates from rest; this leads to a canonical formulation in which the evolution equations are equivalent, for uniform depth, to Whitham's (1967) generalization of the Boussinesq equations, in which dispersion, but not nonlinearity, is assumed to be weak. The further approximation that nonlinearity and dispersion are comparably weak leads to a canonical form of Boussinesq's equations that conserves consistent approximations to energy, momentum (for a level bottom) and potential vorticity.

1. Introduction

We consider here the equations that govern gravity waves on the free surface of a laterally unbounded inviscid fluid of uniform density ρ, ambient depth H and superficial pressure p_e, starting from the single assumption (beyond those of an ideal liquid and Newtonian mechanics) that the fluid moves in vertical columns. This assumption is equivalent to that made by Green & Naghdi (1976, and references cited therein) for the same model and implies the restriction

$$\epsilon \equiv (d/l)^2 \ll 1, \tag{1.1}$$

where d is a depthscale and l is a horizontal lengthscale. This, in turn, implies that dispersion is weak. Our preliminary formulation (§§2 and 3) is for variable depth, but we impose the provisional restriction of uniform depth in §§4–6 and defer the results for variable depth to Appendix C.

Our primary aims are: (i) to derive the Green–Naghdi (GN) equations (see below) and their invariants directly from Hamilton's principle (Green & Naghdi derive them from conservation of energy and invariance under rigid-body translation); (ii) to establish the relation between these equations and those of Boussinesq, of which there are several forms and in which the restriction (1.1) typically is accompanied by the restriction that nonlinearity be comparably weak; (iii) to explore the role of vorticity in the GN and Boussinesq equations. A major advantage of the derivation of the equations of motion from Hamilton's principle is that consistent approximations to energy (including the work done by the external pressure), impulse–momentum (for a level bottom) and potential vorticity are conserved if approximations that preserve the original symmetries are introduced in the action integral and no further approximations are introduced subsequent to the variation of the action (for a more extensive discussion see Salmon 1983).†

We begin with a Lagrangian description of the fluid motion, which leads to the representation of the kinetic and potential energies of the fluid *qua* continuum of particles on which work is done by the external pressure (surface tension could be included through the appropriate addition to the potential energy but is omitted in the interests of simplicity). We then invoke Hamilton's principle for this system to obtain the equation of motion

$$\mathscr{D}\boldsymbol{u}+\boldsymbol{A} = -\boldsymbol{\nabla}(P+g\eta) \quad (\mathscr{D} \equiv \partial_t+\boldsymbol{u}\cdot\boldsymbol{\nabla}), \tag{1.2}$$

where $\boldsymbol{u}$ is the horizontal velocity of a particle, $\mathscr{D}\boldsymbol{u}$ is the corresponding acceleration, $\boldsymbol{A}$ is an auxiliary acceleration that is given by

$$\boldsymbol{A} = \tfrac{1}{3}h^{-1}\boldsymbol{\nabla}(h^2\mathscr{D}^2\eta) \quad (H = \text{constant}) \tag{1.3}$$

for uniform depth and by (C2) for variable depth, $P \equiv p_e/\rho$ is a reduced pressure, η is the free-surface displacement, and $h \equiv \eta+H$ (the depth beneath the displaced surface, but we use *depth* without a modifier to designate the ambient depth H). The corresponding approximation to the equation of continuity is

$$\mathscr{D}h+h\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{1.4}$$

We designate (1.2) and (1.4) as the GN equations; (1.4) is equivalent to (4.27) in Green & Naghdi's 1976 paper; (1.2) can be derived from their (4.28)–(4.30) and is given explicitly by Ertekin (1984).

The acceleration $\boldsymbol{A}$ is $O(\epsilon)$ relative to $\mathscr{D}\boldsymbol{u}$ in (1.2), and its neglect reduces the GN equations to the Airy equations of nonlinear shallow-water theory (Wehausen & Laitone 1960). (But we note that, owing to the presence of $\mathscr{D}^2\eta$ in $\boldsymbol{A}$, the GN equations, in contrast with the Airy equations, do not provide explicit representations of $\boldsymbol{u}_t$ and η_t in terms of $\boldsymbol{u}$ and η and their spatial derivatives.)

It may be inferred from (1.2) and (1.4) that the vorticity $\zeta \equiv \boldsymbol{k}\cdot\boldsymbol{\nabla}\times\boldsymbol{u}$, although initially zero in a flow originating from rest, does not remain so; accordingly, a conventional velocity potential for $\boldsymbol{u}$ does not exist.‡ We find, however, that the *potential vorticity*

$$\Pi \equiv \frac{\zeta+\zeta_*}{h} \tag{1.5}$$

is conserved by particles, where ζ_* is a pseudovorticity that is derived from $\boldsymbol{A}$ and is given by (5.5) for uniform depth and by (C3) for variable depth. It follows that if $\Pi = 0$ initially it remains so, and $\boldsymbol{u}$ then admits a representation of the form $\boldsymbol{u} = \boldsymbol{\nabla}\phi+\chi\boldsymbol{\nabla}\psi$ (see below), in which χ and ψ may be expressed in terms of η and H; however, ϕ, χ and ψ are not uniquely determined (even up to additive functions of t) by this argument.

We determine ϕ (within an additive function of t) through an Eulerian formulation of Hamilton's principle, in which ϕ appears as the Lagrange multiplier of the continuity constraint (1.4) in the variational integrand and the resulting potential

† The inference of integral invariants from symmetries in a variational integral goes back to Noether (1918) and has been prominent, in the present context, in the work of Whitham (1967, 1974). See also Benjamin & Olver (1982).

‡ Green & Naghdi (1976) and Ertekin (1984) overlook this fact and suggest that the constraint $\zeta = 0$ could be imposed *a priori* (presumably by virtue of $\zeta = 0$ at $t = 0$) and hence that $\boldsymbol{u}$ could be derived from a velocity potential; however, they do not pursue this suggestion.

vorticity is zero. We achieve a further simplification by invoking the approximation (which is consistent with the basic approximation of columnar motion) that ζ_* and $\boldsymbol{u}-\nabla\phi$ are $O(\epsilon)$ relative to ζ and $\boldsymbol{u}$, respectively, which leads to the representation

$$\boldsymbol{u} = \nabla\phi+\tfrac{1}{3}h^{-1}\nabla(h^3\nabla^2\phi)+O(\epsilon^2) \quad (H = \text{constant}) \tag{1.6}$$

for the special case of uniform depth and to (C5) for variable depth. The variables ϕ and h then are canonical in Hamilton's sense, and Hamilton's principle implies a pair of evolution equations that are equivalent (for $H =$ constant and $P = 0$) to a generalization of Boussinesq's equations derived by Whitham (1967) from the three-dimensional equations of motion through a variational formulation in which (as in the present development) dispersion, but not nonlinearity, is assumed to be weak, ϕ appears as the velocity potential at the free surface, and $\boldsymbol{u}$ is the depth-averaged horizontal velocity.

The further approximation that nonlinearity is of the same order as dispersion – i.e. that

$$\frac{\eta}{d}, \quad \frac{|\boldsymbol{u}|}{(gd)^{\frac{1}{2}}} = O(\epsilon), \tag{1.7}$$

where d and ϵ are defined as in (1.1), leads to the evolution equations

$$\eta_t+\nabla\cdot(h\nabla\phi)+\tfrac{1}{3}H^3\nabla^4\phi = 0 \tag{1.8a}$$

and

$$\phi_t+\tfrac{1}{2}(\nabla\phi)^2+g\eta+P = 0 \tag{1.8b}$$

in the special case of uniform depth and to (C10) for variable depth. We refer to (1.8*a*,*b*), which are derived from a Hamiltonian formulation in which ϕ and h are canonical variables and which conserve energy and momentum, as the *canonical form of Boussinesq's equations*. In their present form, they appear to be due to Whitham (1967, equation (12)), although equivalent forms have been given by Lin & Clark (1959), Long (1964), Mei & Le Méhauté (1966) and Peregrine (1967).† The corresponding approximation to (1.6) is

$$\boldsymbol{u} = \nabla(\phi+\tfrac{1}{3}H^2\nabla^2\phi), \tag{1.9}$$

by virtue of which the conventional vorticity ζ vanishes in this approximation (but this does not hold for non-uniform depth; see Appendix C).

We conclude that the GN equations are reducible, for flows originating from rest and after approximations that are consistent with the basic assumption of columnar motion, to a generalization of the Boussinesq equations that exactly conserves consistent approximations to all of the invariants of the original equations.

There remains the question of whether either the GN equations or Whitham's and the present generalizations of the Boussinesq equations, in which dispersion is assumed to be weak but full nonlinearity is accommodated (in particular, the boundary conditions at the bottom and free surface are satisfied exactly), are superior (in some definite sense) to the canonical form of the Boussinesq equations, in which dispersion and nonlinearity are assumed to be comparably weak. The balance between dispersion and nonlinearity is intrinsic for the solitary wave, so that the GN and canonical Boussinesq equations should be of comparable validity in the description of such waves and their interactions. The GN description of the solitary wave is identical with that of Rayleigh (1876), the shape of which appears to be inferior to that of Boussinesq (1871) in comparisons with both experiment and more accurate

† The canonical character of (1.8) is emphasized by Miles (1977) and Benjamin (1984).

theoretical calculations for amplitude/depth $\leqslant 0.5$ (Yamada 1958; Miles 1976); on the other hand, it appears that Rayleigh's approximation to the speed of a solitary waves is superior to that of Boussinesq for sufficiently large amplitudes. Ertekin, Webster & Wehausen (1984) compare numerical predictions based on the GN equations and Wu's (1981) formulation of the Boussinesq equations and conclude that 'the Green–Naghdi equations continue to give reasonable-appearing results when Wu's equations have clearly begun to lose validity'.

2. Kinematics

We consider the motion of an incompressible, inviscid fluid of uniform density ρ bounded below by the rigid surface $z = -H(\boldsymbol{x})$ and above by the free surface $z = \eta(\boldsymbol{x}, t)$, where $\boldsymbol{x} \equiv (x, y)$ and z are the Cartesian coordinates of a fluid particle, z is positive up, and $\eta = 0$ in the equilibrium configuration of a level free surface under the action of gravity. We assume that the fluid is in this equilibrium configuration at $t = 0$, choose the corresponding positions of the fluid particles as Lagrangian coordinates, $\boldsymbol{x}_0 \equiv (x_0, y_0)$ and z_0, and pose the description of the fluid motion in the form

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}_0, z_0, \tau), \quad z = z(\boldsymbol{x}_0, z_0, \tau), \quad t = \tau. \tag{2.1a,b,c}$$

The assumptions of incompressibility and uniform density then imply

$$\frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)} = 1. \tag{2.2}$$

The hypothesis that the fluid motion is columnar,

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}_0, \tau), \tag{2.3}$$

reduces the Jacobian (2.2) to

$$\frac{\partial(x, y)}{\partial(x_0, y_0)} \frac{\partial z}{\partial z_0} = 1. \tag{2.4}$$

Integrating (2.4) with respect to z_0 and imposing the condition $z = -H(\boldsymbol{x})$ at $z_0 = -H(\boldsymbol{x}_0)$, i.e. that fluid particles initially on the lower boundary must remain there, we obtain

$$z + H = \frac{\partial(x_0, y_0)}{\partial(x, y)} (z_0 + H_0), \tag{2.5}$$

where, here and subsequently,

$$H \equiv H(\boldsymbol{x}), \quad H_0 \equiv H(\boldsymbol{x}_0). \tag{2.6a,b}$$

Invoking the free-surface condition $z = \eta$ at $z_0 = 0$ in (2.5), introducing

$$h \equiv \eta + H = \frac{\partial(x_0, y_0)}{\partial(x, y)} H_0 \tag{2.7}$$

for the instantaneous depth, and eliminating the Jacobian between (2.5) and (2.7), we obtain

$$\frac{\partial(x, y)}{\partial(x_0, y_0)} = \frac{H_0}{h}, \quad z = \eta + h \frac{z_0}{H_0}. \tag{2.8a,b}$$

The horizontal and vertical components of the particle velocity are given by

$$\boldsymbol{u} \equiv (u, v) = \dot{\boldsymbol{x}}, \quad w = \dot{z} = \dot{\eta} + \dot{h} \frac{z_0}{H_0}, \tag{2.9a,b}$$

where (using subscripts to signify variables fixed during partial differentiation)

$$\dot{f} \equiv \left(\frac{\partial f}{\partial \tau}\right)_{x_0} = \left(\frac{\partial f}{\partial t}\right)_x + \boldsymbol{u}\cdot\boldsymbol{\nabla} f \equiv \mathscr{D}f, \quad \boldsymbol{\nabla} f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right). \tag{2.10 a, b}$$

Differentiating (2.7) with respect to τ,

$$\left(\frac{\partial h}{\partial \tau}\right)_{x_0} = -H_0\left[\frac{\partial(x,y)}{\partial(x_0,y_0)}\right]^{-2}\left[\frac{\partial(\dot{x},y)}{\partial(x_0,y_0)}+\frac{\partial(x,\dot{y})}{\partial(x_0,y_0)}\right] \tag{2.11a}$$

$$= -H_0\frac{\partial(x_0,y_0)}{\partial(x,y)}\left[\frac{\partial(\dot{x},y)}{\partial(x,y)}+\frac{\partial(x,\dot{y})}{\partial(x,y)}\right] = -h\left(\frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{y}}{\partial y}\right), \tag{2.11b}$$

and invoking (2.9) and (2.10), we obtain the equation of continuity in the Eulerian form (1.4).

The vorticity is given by

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla}\times(u,v,w) = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right). \tag{2.12}$$

Its horizontal components are z-dependent in consequence of (2.9*b*); its vertical component is z-independent.

3. The Lagrangian density

We proceed to calculate the kinetic and potential energies of the fluid, regarded as a set of particles with the Lagrangian labels (x_0, y_0, z_0). In transforming from Lagrangian to Eulerian coordinates, we make frequent use of the identities (2.2) and (2.8*a*), which imply

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0 \tag{3.1}$$

for the primitive element of volume and

$$h\,\mathrm{d}x\,\mathrm{d}y = H_0\,\mathrm{d}x_0\,\mathrm{d}y_0 \equiv \mathrm{d}V \tag{3.2}$$

for the columnar element of volume. The (x,y)- and (x_0,y_0)-integrals are (in the absence of lateral boundaries) over the infinite plane.

The kinetic energy is given by

$$T = \tfrac{1}{2}\rho\iint \mathrm{d}x\,\mathrm{d}y\int_{-H}^{\eta}(\boldsymbol{u}^2+w^2)\,\mathrm{d}z \tag{3.3a}$$

$$= \tfrac{1}{2}\rho\iint \mathrm{d}x_0\,\mathrm{d}y_0\int_{-H_0}^{0}\left[\dot{\boldsymbol{x}}^2+\left(\dot{\eta}+\dot{h}\frac{z_0}{H_0}\right)^2\right]\mathrm{d}z_0 \tag{3.3b}$$

$$= \tfrac{1}{2}\rho\iint(\dot{\boldsymbol{x}}^2+\tfrac{1}{3}\dot{h}^2-\dot{h}\dot{\eta}+\dot{\eta}^2)\,\mathrm{d}V. \tag{3.3c}$$

The potential energy, referred to the equilibrium configuration ($z = z_0$), is given by

$$U = \rho g\iint \mathrm{d}x\,\mathrm{d}y\int_{-H}^{\eta}(z-z_0)\,\mathrm{d}z \tag{3.4a}$$

$$= \rho g\iint \mathrm{d}x_0\,\mathrm{d}y_0\int_{-H_0}^{0}\left[\eta+(h-H_0)\frac{z_0}{H_0}\right]\mathrm{d}z_0 \tag{3.4b}$$

$$= \tfrac{1}{2}\rho g\iint(\eta-H+H_0)\,\mathrm{d}V. \tag{3.4c}$$

The work done by the external pressure $p_e(\boldsymbol{x}, t)$ acting on the free surface is

$$W = -\iint p_e \, \mathrm{d}V. \tag{3.5}$$

Combining the preceding results, we obtain the Lagrangian density in the form

$$L \equiv \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}V}(T-U+W) \tag{3.6a}$$

$$= \tfrac{1}{2}[\dot{\boldsymbol{x}}^2 + \tfrac{1}{3}(\dot{\eta}^2 - \dot{H}\dot{\eta} + \dot{H}^2) - g(\eta - H + H_0)] - P \tag{3.6b}$$

$$= \tfrac{1}{2}[\boldsymbol{u}^2 + \tfrac{1}{3}(h\nabla\cdot\boldsymbol{u})^2 + (\boldsymbol{u}\cdot\nabla H)(h\nabla\cdot\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla H)^2 - g(h - 2H + H_0)] - P, \tag{3.6c}$$

where the Eulerian form (3.6c) follows from (3.6b) through the substitutions $\dot{\boldsymbol{x}} = \boldsymbol{u}, \eta = h - H, \dot{h} = -h\nabla\cdot\boldsymbol{u}$, and $\dot{H} \equiv \mathscr{D}H = \boldsymbol{u}\cdot\nabla H$ from (2.9)–(2.11), and

$$P \equiv p_e/\rho \tag{3.7}$$

is a reduced pressure that has the dimensions of specific energy.

4. Hamilton's principle: particle formulation

We now establish the equation of motion through the principle of least action (Hamilton's principle),

$$\delta S \equiv \delta \iiint L \, \mathrm{d}V \, \mathrm{d}\tau = 0, \tag{4.1}$$

where the conventions for variations of the path integral (including integration by parts) are those of classical dynamics (Goldstein 1980).

The analytical manipulations for the general case of non-uniform depth are rather involved; accordingly, we consider the special case of uniform depth here and in §§5 and 6 and then state the corresponding results for variable depth in Appendix C.

The Lagrangian density (3.6b) reduces, for $H = H_0$ and $\dot{H} = 0$, to

$$L = \tfrac{1}{2}(\dot{\boldsymbol{x}}^2 + \tfrac{1}{3}\dot{\eta}^2 - g\eta) - P, \tag{4.2}$$

the substitution of which into (4.1) yields

$$\delta S = -\iiint [\ddot{\boldsymbol{x}}\cdot\delta\boldsymbol{x} + (\tfrac{1}{2}g + \tfrac{1}{3}\ddot{\eta})\,\delta\eta + \nabla P\cdot\delta\boldsymbol{x}]\,\mathrm{d}V\,\mathrm{d}\tau. \tag{4.3}$$

Invoking the identity (Appendix A)

$$\iint F\delta h \, \mathrm{d}V = \iint h^{-1}[\nabla(h^2F)]\cdot\delta\boldsymbol{x}\,\mathrm{d}V, \tag{4.4}$$

substituting $\ddot{\boldsymbol{x}} = \mathscr{D}\boldsymbol{u}$ and $\ddot{\eta} = \mathscr{D}^2\eta$, and introducing

$$\boldsymbol{A} \equiv \tfrac{1}{3}h^{-1}\nabla(h^2\mathscr{D}^2\eta), \tag{4.5}$$

we obtain the Eulerian equation of motion in the form

$$\mathscr{D}\boldsymbol{u} + \boldsymbol{A} = -\nabla(P + g\eta). \tag{4.6}$$

5. Conservation laws

There is a well-known connection between the *symmetry properties* of a Lagrangian and the *conservation laws* of the corresponding dynamical system. The GN equations (1.2) and (1.4) conserve analogues of the total energy, momentum and the potential vorticity on particles because the approximate Lagrangian density (3.6) retains the corresponding symmetry properties of the exact Lagrangian density for three-dimensional flow.

If $P = 0$ conservation of energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = 0, \tag{5.1}$$

and momentum,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \boldsymbol{u}\,\mathrm{d}V = 0, \tag{5.2}$$

correspond (as usual) to the symmetry properties that the Lagrangian density (3.6) is invariant to arbitrary translations of the whole system in time and in space. Confirmation of (5.1) and (5.2) follows from (1.2) and (1.4). If $P \neq 0$ it is necessary only to subtract W (3.5) from $T+V$ in (5.1) and add $\iint P\nabla h\,\mathrm{d}x\,\mathrm{d}y$ to the right-hand side of (5.2).

We define the potential vorticity

$$\Pi \equiv \frac{\zeta+\zeta_*}{h}, \tag{5.3}$$

where
$$\zeta \equiv \boldsymbol{k}\cdot\nabla\times\boldsymbol{u} = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{5.4}$$

is the vertical component of the conventional vorticity,

$$\zeta_* = \frac{1}{3}\frac{\partial(\mathscr{D}\eta,\eta)}{\partial(x,y)} = \tfrac{1}{3}\mathrm{J}(\mathscr{D}\eta,\eta) \tag{5.5}$$

(for uniform depth), and J is an alternative abbreviation for the Jacobian operator in (x,y). The conservation of potential vorticity,

$$\mathscr{D}\Pi = 0, \tag{5.6}$$

then corresponds to the symmetry property that the labels (x_0, y_0) enter the Lagrangian density (4.2) only through the Jacobian $\partial(x_0,y_0)/\partial(x,y)$; see Salmon (1983) for details.

We verify (5.6) by applying the operator $\boldsymbol{k}\cdot\nabla\times$ to (4.6) and invoking (1.4) and (5.4) to obtain

$$h\mathscr{D}(h^{-1}\zeta)+\boldsymbol{k}\cdot\nabla\times\boldsymbol{A} = 0. \tag{5.7}$$

Substituting $\boldsymbol{A}$ from (4.5) and multiplying by h^{-1}, we obtain

$$h^{-1}\boldsymbol{k}\cdot\nabla\times\boldsymbol{A} = \tfrac{1}{3}h^{-3}\frac{\partial(h^2\mathscr{D}^2\eta,h)}{\partial(x,y)} = \tfrac{1}{3}h^{-1}\frac{\partial(\mathscr{D}^2h,h)}{\partial(x,y)} \tag{5.8a}$$

$$= \tfrac{1}{3}H_0^{-1}\frac{\partial(\ddot{\eta},\eta)}{\partial(x_0,y_0)} = \frac{1}{3}\frac{\partial}{\partial\tau}\left[H_0^{-1}\frac{\partial(\dot{\eta},\eta)}{\partial(x_0,y_0)}\right] \tag{5.8b}$$

$$= \tfrac{1}{3}\mathscr{D}\left[h^{-1}\frac{\partial(\mathscr{D}\eta,\eta)}{\partial(x,y)}\right] \equiv \mathscr{D}(h^{-1}\zeta_*), \tag{5.8c}$$

where ζ_* is given by (5.5); (5.6) then follows from (5.3) and (5.7).

The pseudovorticity ζ_* may be recast in the alternative forms

$$\zeta_* = \tfrac{1}{3}n^{-1}\,\mathrm{J}(h^{-n}, h^{n+1}\mathscr{D}h) \tag{5.9a}$$

$$= \tfrac{1}{3}n^{-1}\,\boldsymbol{k}\cdot\nabla\times[h^{-n}\,\nabla(h^{n+1}\mathscr{D}h)] \tag{5.9b}$$

$$= \tfrac{1}{3}\boldsymbol{k}\cdot\nabla\times[(\mathscr{D}h)\,\nabla h], \tag{5.9c}$$

where n is an arbitrary (but non-zero) parameter; (5.9a) reduces to (5.5) for $n=-1$. It follows from (5.6) that if $\zeta+\zeta_*$ is initially zero it remains so, by virtue of which there exist potentials ϕ_n such that $\boldsymbol{u}$ has the one-parameter family of representations

$$\boldsymbol{u} = \nabla\phi_n - \tfrac{1}{3}n^{-1}h^{-n}\nabla(h^{n+1}\mathscr{D}h) \tag{5.10a}$$

$$= \nabla\Phi - \tfrac{1}{3}(\mathscr{D}h)\nabla h, \tag{5.10b}$$

where Φ is a master potential, and

$$\phi_n = \Phi + \tfrac{1}{3}n^{-1}h\mathscr{D}h. \tag{5.11}$$

The potential vorticity Π is closely related to Ertel's (1942) invariant for general three-dimensional flow; see Appendix B.

6. Zero-potential-vorticity flow

The principle of least action may be invoked in an Eulerian reference frame by appending the constraint of continuity and, in the most general case, a Lin constraint (Lin 1963; Seliger & Whitham 1968). The connection between this 'Eulerian' form of Hamilton's principle and the 'particle-mechanics' form used in §4 has been discussed by various authors (see e.g. Bretherton 1970; van Saarloos 1981). We simply assert that the equations obtained by requiring that

$$\delta\iiint[hL+\lambda(\mathscr{D}h+h\nabla\cdot\boldsymbol{u})+\alpha\mathscr{D}\beta]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{6.1}$$

for arbitrary independent variations $\delta\boldsymbol{u}$, δh, $\delta\lambda$, $\delta\alpha$, $\delta\beta$ at fixed (x,y,t) are precisely equivalent to the GN equations; here L is the Lagrangian density (3.6), and λ and α are the Lagrange multipliers of the continuity equation (1.4) and the Lin constraint $\mathscr{D}\beta=0$. If the equation

$$\delta\alpha:\ \mathscr{D}\beta = 0 \tag{6.2}$$

is used to eliminate β from (6.1), then the simplified principle

$$\delta\iiint[hL+\lambda(\mathscr{D}h+h\nabla\cdot\boldsymbol{u})]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{6.3}$$

for variations $\delta\boldsymbol{u}$, δh, $\delta\lambda$ yields equations whose solutions also satisfy the GN equations, but always imply $\Pi=0$.

Considering again the case of uniform depth, for which (3.6c) reduces to

$$L = \tfrac{1}{2}[\boldsymbol{u}^2+\tfrac{1}{3}(h\nabla\cdot\boldsymbol{u})^2-g(h-H)]-P, \tag{6.4}$$

and integrating the constraint term in (6.3) by parts, we obtain

$$\delta\iiint h[\lambda_t+\boldsymbol{u}\cdot\nabla\lambda-\tfrac{1}{2}\boldsymbol{u}^2-\tfrac{1}{6}(h\nabla\cdot\boldsymbol{u})^2+\tfrac{1}{2}g(h-H)+P]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0. \tag{6.5}$$

The variation with respect to λ, which is obtained more directly from (6.3), yields (1.4). The variations with respect to $\boldsymbol{u}$ and h yield

$$\boldsymbol{u} = \nabla\lambda + \tfrac{1}{3}h^{-1}\nabla(h^3\nabla\cdot\boldsymbol{u}), \tag{6.6}$$

for which $\Pi=0$ for any choice of λ, and

$$\lambda_t+\boldsymbol{u}\cdot\nabla\lambda-\tfrac{1}{2}\boldsymbol{u}^2-\tfrac{1}{2}(h\nabla\cdot\boldsymbol{u})^2+g(h-\tfrac{1}{2}H)+P = 0. \tag{6.7}$$

Comparing (6.6) and (5.10a) after invoking $\mathscr{D}h = -h\nabla\cdot\boldsymbol{u}$, we infer that

$$\lambda = \phi_1 + f(t), \tag{6.8}$$

where f is an arbitrary function of t.

We now remark that it is consistent with the columnar approximation (2.3), which implies an $O(\epsilon^2)$ error on the right-hand side of (6.6), wherein the first and second terms are $O(1)$ and $O(\epsilon)$ respectively, to approximate $\boldsymbol{u}$ by $\nabla\lambda = \nabla\phi_1$ in the $O(\epsilon)$ term. We then have (after dropping the subscript 1 from ϕ)

$$\boldsymbol{u} = \nabla\phi + \tfrac{1}{3}h^{-1}\nabla(h^3\nabla^2\phi), \tag{6.9}$$

where, here and subsequently, an $O(\epsilon^2)$ error is implicit. Substituting (6.9) into (6.4) and separating out a pure divergence term, which makes a null contribution to the variational integral, we obtain

$$L = \tfrac{1}{2}[(\nabla\phi)^2 - \tfrac{1}{3}(h\nabla^2\phi)^2 - g(h-H)] + \tfrac{1}{3}h^{-1}\,\nabla\cdot[(h^3\nabla^2\phi)\,\nabla\phi], \tag{6.10}$$

which differs from the exact three-dimensional Lagrangian density for $\Pi = 0$ by $O(\epsilon^2)$ and therefore is as good an approximation thereto as (6.4). The corresponding approximation to (6.5), after choosing $f = -\tfrac{1}{2}gHt$ in (6.8), is

$$\delta\iiint h\,[\phi_t + \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{6}(h\nabla^2\phi)^2 + \tfrac{1}{2}g(h-2H) + P]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0, \tag{6.11}$$

which implies (through the variations $\delta\phi$ and δh) the Boussinesq-like evolution equations

$$h_t + \nabla\cdot(h\nabla\phi) + \tfrac{1}{3}\nabla^2(h^3\nabla^2\phi) = 0 \tag{6.12a}$$

and

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}(h\nabla^2\phi)^2 + P + g\eta = 0. \tag{6.12b}$$

These last approximations also may be obtained by substituting (6.8) and (6.9) into (1.4) and (6.7), but we wish to emphasize that, having adopted the approximation (6.10) to the Lagrangian density, (6.12) follow directly from the principle of least action without further approximation. It also is worth emphasizing that ϕ and h are canonically conjugate variables in (6.11) and (6.12) (cf. Miles 1977).

The evolution equations (6.12) are equivalent (for $P = 0$) to those derived by Whitham (1967, equations (9)–(11)) from a variational principle equivalent to (6.11), in which ϕ appears as the velocity potential at the free surface, and $\boldsymbol{u}$ is the depth-averaged horizontal velocity derived from the z-dependent potential. It follows that, for flows originating from rest (so that the potential vorticity vanishes), the GN equations are reducible to a canonical generalization of the Boussinesq equations in which dispersion is weak but nonlinearity is fully accommodated.

If nonlinearity is assumed to be of the same order as dispersion, i.e. if (1.7) is joined to (1.1), (6.11) may be approximated by

$$\delta\iiint [\eta\phi_t + \tfrac{1}{2}h(\nabla\phi)^2 - \tfrac{1}{6}H^3(\nabla^2\phi)^2 + \tfrac{1}{2}g\eta^2 + P\eta]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0, \tag{6.13}$$

wherein $O(\epsilon^4)$ terms are neglected. The corresponding variational equations are the canonical Boussinesq equations (1.8a,b), whilst the corresponding approximation to (6.9) is (1.9).

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Appendix A. A variational identity

We consider the variational integral

$$\delta I = \iint F\delta h \,\mathrm{d}V, \tag{A 1}$$

where F is any differentiable function of $\boldsymbol{x}$ and τ. Invoking (2.7) for h, substituting

$$\delta h = H_0\,\delta\left[\frac{\partial(x,y)}{\partial(x_0,y_0)}\right]^{-1} = -H_0\left[\frac{\partial(x,y)}{\partial(x_0,y_0)}\right]^{-2}\delta\left[\frac{\partial(x,y)}{\partial(x_0,y_0)}\right] \tag{A 2a}$$

$$= -H_0\left(\frac{h}{H_0}\right)^2\left[\frac{\partial(\delta x,y)}{\partial(x_0,y_0)}+\frac{\partial(x,\delta y)}{\partial(x_0,y_0)}\right] \tag{A 2b}$$

into (A 1), and integrating by parts, we obtain

$$\delta I = \iint H_0^{-1}\left[\frac{\partial(h^2F,y)}{\partial(x_0,y_0)}\delta x+\frac{\partial(x,h^2F)}{\partial(x_0,y_0)}\delta y\right]\mathrm{d}V \tag{A 3a}$$

$$= \iint h^{-1}\left[\frac{\partial(h^2F,y)}{\partial(x,y)}\delta x+\frac{\partial(x,h^2F)}{\partial(x,y)}\delta y\right]\mathrm{d}V \tag{A 3b}$$

$$\equiv \iint h^{-1}\left[\frac{\partial(h^2F)}{\partial x}\delta x+\frac{\partial(h^2F)}{\partial y}\delta y\right]\mathrm{d}V$$

$$= \iint h^{-1}[\boldsymbol{\nabla}(h^2F)]\cdot\delta\boldsymbol{x}\,\mathrm{d}V. \tag{A 3c}$$

Appendix B. Ertel's theorem

In the case of constant fluid density, Ertel's theorem (Ertel 1942; Greenspan 1969) reduces to

$$\mathscr{D}_3\Pi_3 = 0, \tag{B 1}$$

where

$$\Pi_3 \equiv (\boldsymbol{\nabla}_3\times\boldsymbol{u}_3)\cdot\boldsymbol{\nabla}_3\theta, \tag{B 2}$$

$\boldsymbol{u}_3 \equiv (\boldsymbol{u},w)$ is the three-dimensional velocity, $\boldsymbol{\nabla}_3 \equiv (\partial x,\partial y,\partial z)$, $\mathscr{D}_3 \equiv \mathscr{D}+w\,\partial_z$, and θ is any scalar quantity for which

$$\mathscr{D}_3\theta = 0. \tag{B 3}$$

If, as we have assumed, the fluid moves in vertical columns, then the vertical integral of Π_3 is conserved following the columns. That is,

$$\mathscr{D}\Pi = 0, \tag{B 4}$$

where

$$\Pi \equiv H_0^{-1}\int_{-H_0}^{0}\Pi_3\,\mathrm{d}z_0 \tag{B 5a}$$

$$= h^{-1}\int_{-H}^{-H+h}\Pi_3\,\mathrm{d}z. \tag{B 5b}$$

Suppose $\theta = z_0/H_0$. It follows from (2.7)–(2.9) that

$$\theta = h^{-1}(z-h+H) \tag{B 6}$$

and

$$w = -\mathscr{D}H+h^{-1}(z+H)\,\mathscr{D}h. \tag{B 7}$$

For columnar motion, (B 2) reduces to

$$\Pi_3 = \mathrm{J}(\theta, w) + \zeta \frac{\partial \theta}{\partial z}. \tag{B 8}$$

Substituting (B 6) and (B 7) into (B 8), and performing the integration in (B 5*b*), we obtain the equivalent of (5.3) and (C 3).

We emphasize that the derivation of (B 4) from Ertel's theorem is not intended as a substitute for the derivation in §5. Ertel's theorem for three-dimensional flows is not consistent with the columnar approximation and its consequences in the preceding development, and the consistent result (B 4) depends on a special choice of θ (B 6).

Appendix C. Non-uniform depth

Proceeding from (3.6) as in §§4 and 5, but with

$$\delta\eta = \delta h - \nabla H \cdot \delta \boldsymbol{x} \tag{C 1}$$

rather than $\delta\eta = \delta h$, and distinguishing between H and H_0, we obtain the equation of motion (4.6) with (cf. (4.5))

$$\boldsymbol{A} = \tfrac{1}{3}\{\nabla[h\mathscr{D}^2(\eta - \tfrac{1}{2}H)] + (\nabla\eta)\,\mathscr{D}^2(\eta - \tfrac{1}{2}H) + (\nabla H)\,\mathscr{D}^2(H - \tfrac{1}{2}\eta)\} \tag{C 2}$$

and the conservation law (5.6) with (cf. (5.5) and (5.9*a*) with $n = 1$)

$$\zeta_* = \tfrac{1}{3}\mathrm{J}(\mathscr{D}\eta - \tfrac{1}{2}\mathscr{D}H, \eta) + \tfrac{1}{3}\mathrm{J}(\mathscr{D}H - \tfrac{1}{2}\mathscr{D}\eta, H) \tag{C 3a}$$

$$= \tfrac{1}{3}\mathrm{J}(h^{-1}, h^2\mathscr{D}h) - \tfrac{1}{2}\mathrm{J}(h^{-1}, h^2\mathscr{D}H) - \tfrac{1}{2}\mathrm{J}(\mathscr{D}h, H) + \mathrm{J}(\mathscr{D}H, H). \tag{C 3b}$$

The counterparts of (5.10*a*) with $n = 1$ and (5.10*b*) are

$$\boldsymbol{u} = \nabla\phi - \tfrac{1}{3}h^{-1}\,\nabla(h^2\mathscr{D}h) + \tfrac{1}{2}h^{-1}\,\nabla(h^2\mathscr{D}H) + \tfrac{1}{2}(\mathscr{D}h)\,\nabla H - (\mathscr{D}H)\,\nabla H \tag{C 4a}$$

$$= \nabla\Phi - \tfrac{1}{3}(\mathscr{D}\eta - \tfrac{1}{2}\mathscr{D}H)\,\nabla\eta - \tfrac{1}{3}(\mathscr{D}H - \tfrac{1}{2}\mathscr{D}\eta)\,\nabla H, \tag{C 4b}$$

whilst those of (6.9)–(6.12) are

$$\boldsymbol{u} = \nabla\phi + h^{-1}\,\nabla(\tfrac{1}{3}h^3\nabla^2\phi + \tfrac{1}{2}h^2\nabla H\cdot\nabla\phi) - (\tfrac{1}{2}h\nabla^2\phi + \nabla H\cdot\nabla\phi)\,\nabla H, \tag{C 5}$$

$$L = \tfrac{1}{2}[(\nabla\phi)^2 - \tfrac{1}{3}(h\nabla^2\phi)^2 - (h\nabla^2\phi)\,(\nabla H\cdot\nabla\phi) - (\nabla H\cdot\nabla\phi)^2 - g(h - 2H + H_0)] - P + h^{-1}\,\nabla\cdot[(\tfrac{1}{3}h^3\nabla^2\phi + \tfrac{1}{2}h^2\nabla H\cdot\nabla\phi)\,\nabla\phi], \tag{C 6}$$

$$\delta\iiint h[\phi_t + \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{6}(h\nabla^2\phi)^2 - \tfrac{1}{2}(h\nabla^2\phi)\,(\nabla H\cdot\nabla\phi) - \tfrac{1}{2}(\nabla H\cdot\nabla\phi)^2 + \tfrac{1}{2}g(h - 2H) + P]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0, \tag{C 7}$$

$$\eta_t + \nabla\cdot(h\nabla\phi) + \nabla^2(\tfrac{1}{3}h^3\nabla^2\phi + \tfrac{1}{2}h^2\nabla H\cdot\nabla\phi) - \nabla\cdot[(\tfrac{1}{2}h^2\nabla^2\phi + h\nabla H\cdot\nabla\phi)\,\nabla H] = 0, \tag{C 8a}$$

and

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}(h\nabla^2\phi + \nabla H\cdot\nabla\phi)^2 + g\eta + P = 0. \tag{C 8b}$$

The counterparts of the canonical Boussinesq set (6.14), (1.8) and (1.9) are

$$\delta\iiint[\eta\phi_t + \tfrac{1}{2}h(\nabla\phi)^2 - \tfrac{1}{6}H^3(\nabla^2\phi)^2 - \tfrac{1}{2}(H^2\nabla^2\phi)\,(\nabla H\cdot\nabla\phi) - \tfrac{1}{2}H(\nabla H\cdot\nabla\phi)^2 + \tfrac{1}{2}g\eta^2 + P\eta]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0, \tag{C 9}$$

$$\eta_t + \nabla\cdot(h\nabla\phi) + \nabla\cdot[\tfrac{1}{2}H^2\nabla\nabla\cdot(H\nabla\phi) - \tfrac{1}{6}H^3\nabla\nabla^2\phi] = 0, \quad \text{(C 10}a\text{)}$$

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\eta + P = 0, \quad \text{(C 10}b\text{)}$$

$$\boldsymbol{u} = \nabla\phi + \tfrac{1}{2}H\nabla\nabla\cdot(H\nabla\phi) - \tfrac{1}{6}H^2\nabla\nabla^2\phi. \quad \text{(C 11)}$$

Substituting $\boldsymbol{u}' \equiv \nabla\phi$ into (C 10a) and the gradient of (C 10b), we recover one of Peregrine's (1967, equations (16)–(18)) forms of the Boussinesq equations for variable depth. Substituting (C 11) into (C 10a,b), we recover Peregrine's alternative form of the Boussinesq equations (*ibid.*, equations (13) and (14), wherein $\bar{\boldsymbol{u}} = \boldsymbol{u}$ above). We remark that the vorticity ζ is *not* conserved here, but that $\zeta' \equiv \boldsymbol{k}\cdot\nabla\times\boldsymbol{u}' \equiv 0$ (cf. Peregrine's equation (15)). See also Wu (1981), wherein $\bar{\phi} = \phi + \tfrac{1}{3}H^2\nabla^2\phi + \tfrac{1}{2}H\nabla H\cdot\nabla\phi$ in the present notation.

Linear approximation

Neglecting the second-order terms in (C 8b), we obtain

$$\eta = -g^{-1}(\phi_t + P), \quad \text{(C 12)}$$

the substitution of which, together with $h \approx H$, into (C 8a) yields, after some reduction,

$$g\nabla\cdot\{H\nabla\phi + \tfrac{1}{2}H^2\nabla\nabla\cdot(H\nabla\phi) - \tfrac{1}{6}H^3\nabla\nabla^2\phi\} = \phi_{tt} + P_t, \quad \text{(C 13)}$$

which is equivalent, for $P = 0$, to an earlier result of Miles (1985).

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