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Semigeostrophic theory as a Dirac-bracket projection

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This paper presents a general method for deriving approximate dynamical equations that satisfy a prescribed constraint typically chosen to filter out unwanted high-frequency motions. The approximate equations take a simple general form in arbitrary phase–space coordinates. The family of *semigeostrophic* equations for rapidly rotating flow derived by Salmon (1983, 1985) fits this general form when the chosen constraint is geostrophic balance. More precisely, the semigeostrophic equations are equivalent to a Dirac-bracket projection of the exact Hamiltonian fluid dynamics onto the phase–space manifold corresponding to geostrophically balanced states. The more widely used *quasi-geostrophic* equations do not fit the general form, and are instead equivalent to a *metric* projection of the exact dynamics on to the same geostrophic manifold. The metric, which corresponds to the Hamiltonian of the linearized dynamics, is an artificial component of the theory, and its presence explains why the quasi-geostrophic equations are valid only near a state with flat isopycnals.

1. Introduction

This paper presents a method for deriving approximate dynamical equations that satisfy a prescribed constraint typically chosen to filter out high-frequency waves. The method is illustrated by application to the equations for a shallow homogeneous fluid in rotating coordinates. For simplicity, we suppose that the fluid is unbounded and quiescent at infinity, and that the Coriolis parameter is constant. This shallow-water system is a paradigm for the primitive equations of meteorology and oceanography. However, the methods proposed here are very general, and will be useful in other applications.

The quasi-geostrophic equations are the best-known approximation to the general equations for rotating incompressible flow. Both the quasi-geostrophic equations and the semigeostrophic equations discussed below filter out inertia-gravity waves. These waves are unimportant contributors to weather and to the large-scale ocean circulation, but their presence attaches a severe penalty to the use of the primitive equations as the basis for numerical models. Unfortunately, the quasi-geostrophic equations apply to flows in which the fluid depth (or the mass density, in the case of a continuous stratification) departs only slightly from a prescribed, horizontally uniform state. As we show below, this restriction is not needed to filter out inertia-gravity waves, and is an artifact of the method by which the quasi-geostrophic equations are derived.

The semigeostrophic equations (Hoskins 1975) apply to nearly geostrophic flow in which the free surface (or isopycnals) may be far from level. The semigeostrophic equations, which take a simple form in cleverly chosen 'geostrophic coordinates', have been widely used in meteorology. For a recent review, refer to Hoskins, McIntyre & Robertson (1985). However, Hoskins's form of the semigeostrophic equations conserves a form of the potential vorticity only if the Coriolis parameter is a constant.

By approximations to Hamilton's principle, Salmon (1983, 1985) derived generalized semigeostrophic equations that apply to the geophysically important case of a spatially varying Coriolis parameter. The generalized semigeostrophic equations automatically conserve analogues of the exact invariants of the motion, because the approximations do not disturb the corresponding symmetry properties of the Hamiltonian. Moreover, the Hamiltonian derivation motivates a transformation to canonical variables, and these turn out to be the 'geostrophic coordinates'. However, despite these advantages, the specific procedure followed by Salmon (1983, 1985) seemed somewhat *ad hoc*.

In this paper we show that the methods of Salmon (1983, 1985) are not really *ad hoc*, and have an illuminating geometrical interpretation that permits generalization. More precisely, we show that the generalized semigeostrophic equations are equivalent to a Dirac-bracket projection of the exact shallow-water dynamics onto the phase-space manifold corresponding to geostrophic balance. The semigeostrophic equations take a simple general form in coordinate-free notation. The quasi-geostrophic equations do not fit this general form, and are instead equivalent to a *metric* projection onto the same geostrophic manifold. The metric, which corresponds to the Hamiltonian of the linearized shallow-water equations, is an artificial component of the theory, and its presence explains why the quasi-geostrophic equations are valid only near a state with flat isopycnals.

The methods of Salmon (1983, 1985) and the present paper extend to fully three-dimensional, stratified flow. However, the fundamental ideas are best explained by application to the simpler case of a shallow homogeneous fluid in a uniformly rotating reference frame.

2. Quasi-geostrophic equations as a Galerkin approximation

The equations for a shallow homogeneous fluid in coordinates rotating at constant angular velocity $\frac{1}{2}f$ about the vertical, are

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{f}\times\boldsymbol{u} - g\,\nabla h, \tag{2.1}$$

and

$$\frac{\partial h}{\partial t} = -\nabla\cdot(\boldsymbol{u}h). \tag{2.2}$$

Here, $\boldsymbol{u} = (u,v)(x,y,t)$ is the horizontal velocity of the vertical fluid column at location (x,y) and time t, $h(x,y,t)$ is the fluid depth, g is the gravity, $\nabla = (\partial_x, \partial_y)$, and $\boldsymbol{f} = f\boldsymbol{k}$ where $\boldsymbol{k}$ is the vertical unit vector. For simplicity, we suppose that the fluid is unbounded and quiescent at infinity. Then the energy

$$H = \frac{1}{2}\iint \mathrm{d}\boldsymbol{x}\, h[\boldsymbol{u}\cdot\boldsymbol{u} + gh] \tag{2.3}$$

is exactly conserved. Let h_0 be the mean depth of the fluid. The quasi-geostrophic approximation to (2.1), (2.2) is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)[\nabla^2\eta - (f^2/gh_0)\,\eta] = 0, \tag{2.4}$$

where now
$$f \times u = -g\nabla\eta, \tag{2.5}$$
and
$$\eta \equiv h - h_0, \tag{2.6}$$
is the free-surface elevation. It is easy to see that (2.4)–(2.6) is a close approximation to the exact potential vorticity equation,

$$\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)q = 0, \quad q = \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)/(h_0+\eta) \tag{2.7}$$

provided that the exact flow is nearly geostrophic, and that the free-surface elevation is small,
$$|\eta| \ll h_0. \tag{2.8}$$
The condition (2.8) is the previously mentioned artificial restriction.

Leith (1980) has shown that the quasi-geostrophic approximation is really a Galerkin approximation, in which we first expand the exact flow in the complete set of modes of the linearized equations, and then discard both the modes corresponding to inertia-gravity waves and the projections of all time-derivatives onto these same modes. In the remainder of this section, we review Leith's procedure for the shallow-water example. In the following section, we reformulate it in a coordinate-free, geometric notation for a general dynamical system. The general formulation pinpoints the fundamental flaw in the quasi-geostrophic approximation, and it forms an interesting contrast to later results.

Let the variables be rescaled so that $f = g = h_0 = 1$, and let

$$u(x,t) = \iint \mathrm{d}k\, u(k,t)\, \mathrm{e}^{\mathrm{i}k\cdot x} \quad (k = (k,l)), \tag{2.9}$$

and $\eta(x,t)$ be represented by spatial Fourier transforms in the usual way. Then if the shallow-water equations (2.1), (2.2) are linearized about the state $u(x) = 0$ and $h(x) = h_0$, the resulting equations can be written in the non-dimensional form

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi = H\psi, \tag{2.10}$$

where ψ is the column vector
$$\psi(k,t) = \begin{pmatrix} u(k,t) \\ v(k,t) \\ \eta(k,t) \end{pmatrix}, \tag{2.11}$$

and
$$H = \begin{pmatrix} 0 & \mathrm{i} & k \\ -\mathrm{i} & 0 & l \\ k & l & 0 \end{pmatrix}, \tag{2.12}$$

is a Hermitian matrix. The linearized equations exactly conserve the (non-dimensional) energy
$$H_0 = \frac{1}{2}\iint \mathrm{d}x\,[u\cdot u+\eta^2]. \tag{2.13}$$

The general solution to (2.10)–(2.12) is

$$\psi(k,t) = \sum_{n=0,1,2} C_n(k)\, e_n(k) \exp(-\mathrm{i}\omega_n(k)\,t), \tag{2.14}$$

where ω_n and $\boldsymbol{e}_n$ are the eigenvalues and eigenvectors of $\boldsymbol{H}$. Since $\boldsymbol{H}$ is Hermitian, the ω_n are real, and

$$\boldsymbol{e}_n^+ \boldsymbol{e}_m \equiv \boldsymbol{e}_n \cdot \boldsymbol{e}_m = \delta_{nm} \boldsymbol{e}_n \cdot \boldsymbol{e}_n, \tag{2.15}$$

where $^+$ denotes the transpose conjugate. We find that

$$\omega_0 = 0, \quad \omega_1, \omega_2 = \pm\{1+k^2+l^2\}^{\frac{1}{2}} \tag{2.16}$$

and

$$\boldsymbol{e}_0 = \begin{pmatrix} -\mathrm{i}l \\ \mathrm{i}k \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_{1,2} = \begin{pmatrix} k\omega+\mathrm{i}l \\ l\omega-\mathrm{i}k \\ k^2+l^2 \end{pmatrix}. \tag{2.17}$$

The zeroth mode corresponds to steady geostrophic motion, while the first and second modes correspond to inertia-gravity waves moving in opposite directions.

Now regard $\boldsymbol{\psi}(\boldsymbol{k}, t)$ as the state vector for the general nonlinear shallow-water system. The quasi-geostrophic approximation (2.4)–(2.6) is equivalent to the projections

$$\boldsymbol{e}_1 \cdot \boldsymbol{\psi} = \boldsymbol{e}_2 \cdot \boldsymbol{\psi} = 0 \tag{2.18}$$

and

$$\boldsymbol{e}_1 \cdot \mathrm{d}\boldsymbol{\psi}/\mathrm{d}t = \boldsymbol{e}_2 \cdot \mathrm{d}\boldsymbol{\psi}/\mathrm{d}t = 0. \tag{2.19}$$

Equations (2.18), which define the 'slow manifold', are simply the Fourier transforms of the equations (2.5) for geostrophic balance. Equations (2.19) can be combined with (2.18) and the general shallow-water equations (2.1)–(2.2) to yield the quasi-geostrophic potential vorticity equation (2.4).

3. Quasi-geostrophic approximation in geometric notation

We now repeat the foregoing derivation in a coordinate-free notation that applies to general Hamiltonian systems. This geometric formulation pinpoints the fundamental flaw in the quasi-geostrophic approximation.

Let $(z^i(t), i = 1 \text{ to } N)$ be coordinates in the N-dimensional phase space with dynamics

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = J^{ij} \frac{\partial H(z)}{\partial z^j}. \tag{3.1}$$

Repeated indices are summed. Here $H(z)$ is the Hamiltonian and $J^{ij}(z)$ is an antisymmetric contravariant tensor with the Jacobi property,

$$J^{im} \frac{\partial J^{jk}}{\partial z^m} + J^{jm} \frac{\partial J^{ki}}{\partial z^m} + J^{km} \frac{\partial J^{ij}}{\partial z^m} = 0. \tag{3.2}$$

The tensor J^{ij} can be singular or not; if J^{ij} is non-singular, then canonical coordinates exist. If J^{ij} is singular with corank K, then there exist K independent *Casimir* functions $(C_{(k)}, k = 1 \text{ to } K)$ such that

$$J^{ij} \frac{\partial C_{(k)}}{\partial z^j} = 0 \quad \text{at every } z. \tag{3.3}$$

Let

$$\{F, G\} \equiv \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j}, \tag{3.4}$$

be the Poisson bracket, defined for any two phase functions $F(z)$ and $G(z)$. Then

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}, \tag{3.5}$$

where H is the Hamiltonian. Thus the whole dynamics is specified by the scalar $H(z)$ and the bilinear operator $\{\ ,\ \}$.

The shallow-water equations fit this general Hamiltonian form with infinite N. Two precise forms are convenient. In the first, the coordinates z^i are the velocities $\boldsymbol{u}(\boldsymbol{a})$ and locations $\boldsymbol{x}(\boldsymbol{a})$ of the fluid particle identified by Lagrangian labels $\boldsymbol{a} = (a, b)$. The index i corresponds to $\boldsymbol{a}$, and time arguments have been suppressed. The labels are assigned so that $\mathrm{d}\boldsymbol{a} = h\,\mathrm{d}\boldsymbol{x}$ at all times.

In the second form, the coordinates are the velocities $\boldsymbol{u}(\boldsymbol{x})$ and fluid depth $h(\boldsymbol{x})$ at location $\boldsymbol{x}$.

The Hamiltonian in the two forms is

$$H = \frac{1}{2}\iint \mathrm{d}\boldsymbol{a}\left[\boldsymbol{u}\cdot\boldsymbol{u}+g\frac{\partial(a,b)}{\partial(x,y)}\right] = \frac{1}{2}\iint \mathrm{d}\boldsymbol{x}\, h[\boldsymbol{u}\cdot\boldsymbol{u}+gh], \tag{3.6}$$

and the corresponding Poisson brackets are

$$\{F,G\} = \iint \mathrm{d}\boldsymbol{a}\left[\frac{\delta F}{\delta \boldsymbol{x}}\cdot\frac{\delta G}{\delta \boldsymbol{u}}-\frac{\delta F}{\delta \boldsymbol{u}}\cdot\frac{\delta G}{\delta \boldsymbol{x}}+f\frac{\delta F}{\delta u_i}\epsilon_{ij}\frac{\delta G}{\delta u_j}\right] \tag{3.7a}$$

$$= \iint \mathrm{d}\boldsymbol{x}\left[\boldsymbol{\nabla}\cdot\left(\frac{\delta F}{\delta \boldsymbol{u}}\right)\frac{\delta G}{\delta h}-\frac{\delta F}{\delta h}\boldsymbol{\nabla}\cdot\left(\frac{\delta G}{\delta \boldsymbol{u}}\right)+q\frac{\delta F}{\delta u_i}\epsilon_{ij}\frac{\delta G}{\delta u_j}\right], \tag{3.7b}$$

where q is the potential vorticity defined in (2.7) and ϵ_{ij} is the permutation symbol. For a more thorough explanation of (3.1)–(3.7) refer to Salmon (1988).

Now let z_0 be the stable fixed point corresponding to a state of rest and minimum energy. Since z_0 is a fixed point, it follows from (3.1) and (3.3) that

$$\frac{\partial H}{\partial z^i} = -\lambda_{(k)}\frac{\partial C_{(k)}}{\partial z^i} \quad \text{at } z = z_0, \tag{3.8}$$

where $\lambda_{(k)}$ are K constants, and $C_{(k)}$ are the K Casimirs defined above. (Note that $K = 0$ if J^{ij} is non-singular.) By (3.3) and (3.8), the linearization of (3.1) about z_0 can be written

$$\mathrm{i}\frac{\mathrm{d}\Delta z^i}{\mathrm{d}t} = H^i_k\,\Delta z^k, \tag{3.9}$$

where

$$\Delta z^i \equiv z^i - z^i_0, \tag{3.10}$$

$$H^i_k \equiv \mathrm{i}[J^{ij}]_0\, g_{jk}, \tag{3.11}$$

$$g_{jk} \equiv \left[\frac{\partial^2 I}{\partial z^j\,\partial z^k}\right]_0, \tag{3.12}$$

$$I \equiv H + \lambda_{(k)} C_{(k)}, \tag{3.13}$$

and $[\ ]_0$ means that the enclosed quantity is evaluated at z_0. The linearized dynamics (3.9) conserves the energy

$$H_0 = \tfrac{1}{2}g_{ij}\,\Delta z^i\,\Delta z^j. \tag{3.14}$$

Since z_0 is, by assumption, a state of minimum energy, the quadratic form (3.14) is positive definite. Then, since the g_{ij} are constants, there exist Cartesian coordinates in which the *metric* $g_{ij} = \delta_{ij}$ and the linearized energy is

$$H_0 = \tfrac{1}{2}\Delta z^i\,\Delta z^i. \tag{3.15}$$

Equation (2.13) is a case of (3.15). In the Cartesian coordinates H^i_k is Hermitian. It follows that the eigenvalues

$$\omega_{(1)} < \omega_{(2)} < \ldots < \omega_{(N)}, \tag{3.16}$$

of H^i_k, which are the same in all coordinates, are real, and that the corresponding eigenvectors

$$e_{(1)}, e_{(2)}, \ldots, e_{(N)}, \tag{3.17}$$

are orthogonal,

$$(e^i_{(n)})^* g_{ij}\, e^j_{(m)} \equiv \langle e_{(n)}, e_{(m)}\rangle = \delta_{nm}\langle e_{(n)}, e_{(n)}\rangle, \tag{3.18}$$

where * denotes the complex conjugate.

For the moment, we work in Cartesian coordinates. Let $\{\omega_{(s)}\}$ be the smallest $M < N$ eigenvalues of H^i_k and let $\{e_{(s)}\}$ be the corresponding 'slow eigenstates'. Let $\{\omega_{(F)}\}$ and $\{e_{(F)}\}$ be the remaining $N-M$ eigenvalues and corresponding 'fast eigenstates'. Let

$$z = z_0 + c_{(s)}\, e_{(s)} \tag{3.19}$$

(where $c_{(s)}$ are M constants) be an arbitrary point on the *slow manifold*, defined as the intersection of the $N-M$ hyperplanes

$$\langle e_{(F)}, (z-z_0)\rangle \equiv \phi_{(F)}(z) = 0. \tag{3.20}$$

In these Cartesian coordinates

$$e_{(F)i} = \frac{\partial \phi^*_{(F)}}{\partial z^i}. \tag{3.21}$$

The approximate dynamics on the slow manifold are now defined to be

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = \{z^i, H\} + \mu_{(F)}\, e^i_{(F)}, \tag{3.22}$$

where the $N-M$ coefficients $\mu_{(F)}$ are determined by the requirement that the system remain on the slow manifold defined by (3.20). That is,

$$\begin{aligned} 0 = \frac{\mathrm{d}\phi_{(F)}}{\mathrm{d}t} &= \frac{\partial \phi_{(F)}}{\partial z^i}[\{z^i, H\} + \mu_{(F')}\, e^i_{(F')}] \\ &= e^*_{(F)i}[\{z^i, H\} + \mu_{(F')}\, e^i_{(F')}] \\ &= \langle e_{(F)}, \{z, H\}\rangle + \mu_{(F)}\langle e_{(F)}, e_{(F)}\rangle, \end{aligned} \tag{3.23}$$

with no summation on F in the final expression. Here, $\{z, H\}$ is the vector with components $\{z^i, H\}$. Thus (3.22) is equivalent to

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = \{z^i, H\} - \langle \hat{e}_{(F)}, \{z, H\}\rangle\, \hat{e}_{(F)}, \tag{3.24}$$

where $\hat{e}_{(F)}$ is the unit vector in the direction of $e_{(F)}$. The right-hand side of (3.24) is simply the metric projection of the exact expression $\{z^i, H\}$ for $\mathrm{d}z^i/\mathrm{d}t$ onto the hyperplanes defined by (3.20).

We now write our final results in a form valid in arbitrary phase coordinates z^i. This covariant formulation reveals the fundamental ingredients of the theory. In a general Hamiltonian phase space, let

$$\phi_{(l)}(z) = 0 \quad (l = 1 \text{ to } N-M), \tag{3.25}$$

be the equations for an arbitrarily prescribed M-dimensional slow manifold. We assume, with no loss in generality, that the constraint functions are real. (The equations (3.20) can always be rearranged so that this is so.) The constraints (3.25) might be obtained, as above, from an analysis of the linearized dynamics, but we now make no restrictions. Let $F(z)$ be an arbitrary phase function. Then the approximate evolution of $F(z)$ on the slow manifold (3.25) is defined to be

$$\frac{\mathrm{d}F(z)}{\mathrm{d}t} = \{F, H\} + \mu_{(l)}\langle \nabla F, \nabla \phi_{(l)}\rangle, \tag{3.26}$$

where
$$0 = \{\phi_{(l)}, H\} + \mu_{(m)}\langle \nabla \phi_{(l)}, \nabla \phi_{(m)}\rangle, \tag{3.27}$$

and
$$\langle \nabla F, \nabla G\rangle \equiv \frac{\partial F}{\partial z^i} g^{ij} \frac{\partial G}{\partial z^j}, \tag{3.28}$$

and the metric g_{ij} is defined by (3.12). Equations (3.26)–(3.28) (with $F(z) = z^i$) express the same physics as (3.22), (3.23) in general non-Cartesian coordinates.

The covariant formulation (3.26)–(3.28) reveals that the quasi-geostrophic approximation has the following fundamental ingredients: the Hamiltonian $H(z)$ and Poisson brackets $\{\ ,\ \}$ of the exact dynamics, the constraint functions $\phi_{(l)}(z)$ defining the slow manifold, and the metric g_{ij} obtained from the Hamiltonian (3.14) for the linearized dynamics.

The metric g_{ij} is the objectionable feature of the theory. Phase space has no natural metric, and the global imposition of the metric (3.12), which depends only on the local dynamics near z_0, is extremely artificial. It is therefore unsurprising that the resulting approximation becomes inaccurate for z far from z_0.

Two earlier papers (Salmon 1983, 1985) derived semigeostrophic approximations to the shallow-water equations that are free of the artificial restriction on the quasi-geostrophic approximation that z be close to z_0. (The term *semigeostrophic* has a precise meaning in meteorology, but here I use it to denote a general class that includes the conventional semigeostrophic approximation.) Although the methods used to derive these approximations were somewhat *ad hoc*, they were firmly based on Hamiltonian theory. In §4 we show that, when the semigeostrophic approximation of Salmon (1985, §2) is written in a geometrical notation analogous to (3.26), (3.27), the result is

$$\frac{\mathrm{d}F(z)}{\mathrm{d}t} = \{F, H\} + \mu_{(l)}\{F, \phi_{(l)}\}, \tag{3.29}$$

where
$$0 = \{\phi_{(l)}, H\} + \mu_{(m)}\{\phi_{(l)}, \phi_{(m)}\}, \tag{3.30}$$

and the constraint equations (3.25) are the same equations of geostrophic balance as in the quasi-geostrophic theory. Equations (3.29), (3.30) differ from (3.26), (3.27) only in that the Poisson bracket $\{\ ,\ \}$ replaces the metric product $\langle\ ,\ \rangle$.

That is, the quasi-geostrophic and semigeostrophic approximations take the respective forms

$$\frac{\mathrm{d}F(z)}{\mathrm{d}t} = \left[\frac{\mathrm{d}F(z)}{\mathrm{d}t}\right]_{\text{exact}} + \mu_{(l)} \frac{\partial F}{\partial z^i} g^{ij} \frac{\partial \phi_{(l)}}{\partial z^j} \tag{3.31}$$

and
$$\frac{\mathrm{d}F(z)}{\mathrm{d}t} = \left[\frac{\mathrm{d}F(z)}{\mathrm{d}t}\right]_{\text{exact}} + \mu_{(l)} \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial \phi_{(l)}}{\partial z^j}, \tag{3.32}$$

where, in both cases, the $N-M$ coefficients $\mu_{(l)}$ are determined by the $N-M$ requirements

$$\frac{\mathrm{d}\phi_{(l)}}{\mathrm{d}t} = 0 \tag{3.33}$$

that the phase-space trajectories of the approximate dynamics remain on the slow manifold (3.25). Equation (3.32) differs from (3.31) only in that the symplectic tensor J^{ij} replaces the artificial metric g^{ij}.

The general equations (3.29), (3.30) were derived by Dirac (1950, 1958) in a different context. The right-hand side of (3.29) is called the *Dirac bracket* of F and H, and an extensive literature exists. (My use of the term 'Dirac-bracket *projection*' is unconventional, and is designed to emphasize the analogy between (3.31) and (3.32)). The relevant part of Dirac's theory will be sketched in §4. In §5, we turn to the question of how best to choose the constraint functions $\phi_{(l)}(z)$ that define the slow manifold. It turns out that many choices for the constraint functions filter out high-frequency motion. The optimum choice is that for which the equations (3.29), (3.30) take the simplest mathematical form in coordinates that cover the slow manifold.

4. Semigeostrophic approximation in geometric notation

As shown by Salmon (1983, 1985), the shallow-water equations (2.1)–(2.2) result from Hamilton's principle in the form

$$0 = \delta \int dt \left\{ \iint d\boldsymbol{a}\, [u_i(\boldsymbol{a},t) - \tfrac{1}{2} f \epsilon_{ij} x_j(\boldsymbol{a},t)] \frac{\partial x_i(\boldsymbol{a},t)}{\partial t} - H \right\}, \tag{4.1}$$

for variations $\delta u_i(\boldsymbol{a},t), \delta x_i(\boldsymbol{a},t)$ in the velocities and locations of marked fluid particles. Here, u_i is the velocity in the x_i-direction, and H is the Hamiltonian (3.6*a*). The principle (4.1) is analogous to

$$0 = \delta \int dt \left[p_i \frac{dq_i}{dt} - H(p,q) \right], \tag{4.2}$$

for variations $\delta p_i(t), \delta q_i(t)$ in general canonical coordinates. In (4.2) the subscript i is analogous to $\boldsymbol{a}$ *and* the directional subscript on velocity or location.

Salmon (1985, §2) derived a semigeostrophic approximation by replacing $\boldsymbol{u}(\boldsymbol{a},t)$ in (4.1) by its geostrophic value (2.5), which is a functional of $\boldsymbol{x}(\boldsymbol{a},t)$. In the simpler notation of (4.2), this replacement takes the form:

$$p_i = \psi_i(q), \tag{4.3}$$

where the ψ_i are prescribed functions of all the q_j. The semigeostrophic equations result from

$$0 = \delta \int dt \left[\psi_i(q) \frac{dq_i}{dt} - H(\psi(q), q) \right], \tag{4.4}$$

for variations $\delta q_i(t)$. We find that

$$\left(\frac{\partial \psi_j}{\partial q_i} - \frac{\partial \psi_i}{\partial q_j} \right) \frac{dq_j}{dt} \;`=\!'\; \frac{\partial H}{\partial p_j} \frac{\partial \psi_j}{\partial q_i} + \frac{\partial H}{\partial q_i}, \tag{4.5}$$

where the symbol '=' denotes the equality holds only on the slow manifold (4.3).

We next rewrite the constraints (4.3) in the form

$$\phi_i(p,q) \equiv p_i - \psi_i(q) = 0, \tag{4.6}$$

and define the Poisson bracket

$$\{F,G\} = \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i}. \tag{4.7}$$

This definition agrees with (3.4) in the case of canonical coordinates. After some manipulation, (4.5) is equivalent to

$$\{\phi_i, \phi_j\}\left(\frac{\mathrm{d}q_j}{\mathrm{d}t} - \{q_j, H\}\right) `=' \{H, \phi_i\}. \tag{4.8}$$

If $\{\phi_i, \phi_j\}$ is non-singular, then $\mathrm{d}q_i/\mathrm{d}t$ can be found. For non-singular $\{\phi_i, \phi_j\}$, (4.8) is equivalent to (3.29)–(3.30).

We now take a broader view. Let

$$\phi_{(l)}(z) = 0 \tag{4.9}$$

be any $N-M$ independent constraints defining an arbitrary M-dimensional manifold in phase space. We require that the exact Langrangian in (4.2) be stationary subject to the constraints (4.9). That is,

$$0 = \delta \int \mathrm{d}t \left[p_i \frac{\mathrm{d}q_i}{\mathrm{d}t} - H - \mu_{(l)}\phi_{(l)}\right], \tag{4.10}$$

for variations $\delta z^i(t)$ and $\delta\mu_{(l)}(t)$, where $\mu_{(l)}$ are the Lagrangian multipliers corresponding to (4.9). These variations yield

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = \{z^i, H\} + \mu_{(l)}\{z^i, \phi_{(l)}\}, \tag{4.11}$$

and (4.9). Let $F(z)$ be an arbitrary phase function. By (4.11), the change in $F(z)$ following a phase point moving on the manifold (4.9) is given by

$$\frac{\mathrm{d}F(z)}{\mathrm{d}t} `=' \{F, H\} + \mu_{(l)}\{F, \phi_{(l)}\} `=' \{F, H_T\}, \tag{4.12}$$

where $H_T \equiv H + \mu_{(l)}\phi_{(l)}$, and the $\mu_{(l)}$ must satisfy

$$\frac{\mathrm{d}\phi_{(l)}}{\mathrm{d}t} `=' 0 `=' \{\phi_{(l)}, H\} + \mu_{(m)}\{\phi_{(l)}, \phi_{(m)}\}, \tag{4.13}$$

so that the trajectory (4.11) remains on (4.9). For the moment, $\{\phi_{(l)}, \phi_{(m)}\}$ can be singular.

Dirac (1950, 1958) derived (4.12)–(4.13) as the *exact* equations for a dynamical system with a singular Lagrangian. A Lagrangian is called singular if the defining equations

$$p_i = \frac{\partial L}{\partial(\mathrm{d}q_i/\mathrm{d}t)}, \tag{4.14}$$

cannot be solved for all of the $\mathrm{d}q_i/\mathrm{d}t$ in terms of p and q. If the Lagrangian is singular, then, as shown by Dirac, constraints exist that restrict p and q to a manifold in phase space. The determination of these constraints is a non-trivial matter. For a thorough discussion of the theory, refer to Dirac (1964), Hanson, Regge & Teitelboim (1976), Sundermeyer (1982), and Sudarshan & Mukunda (1983). However, in the present context, the constraints (4.9) are *prescribed* and constitute an approximation to the exact dynamics. Although much of Dirac's theory can be taken over, the situation is actually somewhat simpler.

Returning to our problem, let the $\phi_{(l)}(z)$ be a trial set of constraint functions defining a candidate slow manifold. If $\{\phi_{(l)}, \phi_{(m)}\}$ is non-singular everywhere on (4.9), then (4.12)–(4.13) define a unique trajectory through every point on the slow manifold.

Now suppose that $\{\phi_{(l)}, \phi_{(m)}\}$ is singular. Let $\boldsymbol{e}_j(z)$ be the jth null eigenvector,

$$\{\phi_{(l)}, \phi_{(m)}\}\, e_{(m)j} = 0. \tag{4.15}$$

Then (4.13) implies that

$$\chi_j(z) \equiv \{H, \phi_{(l)}\}\, e_{(l)j} \text{ `='}\, 0, \tag{4.16}$$

for each j. If the additional constraints (4.16) are not automatically satisfied on (4.9), we add those $\chi_j(z)$ that are independent of the $\phi_{(l)}(z)$ and each other to the set of trial constraints (4.9) and begin anew. The new slow manifold has a lower dimension by the number of independent χ_j. We continue this process until no new independent χ_j turn up. In a finite-dimensional phase space, this process must terminate. In an infinite-dimensional phase space, we *require* that it terminates, and call the resulting $\phi_{(l)}(z)$ consistent.

If the $\phi_{(l)}(z)$ are consistent, then $\{\phi_{(l)}, \phi_{(m)}\}$ may be singular, but (4.16) are automatically satisfied. In this case, let $U_{(m)}(z)$ be a particular solution of (4.13). The general solution is

$$\mu_{(m)} = U_{(m)} + w_j\, e_{(m)j}, \tag{4.17}$$

where $w_j(t)$ are arbitrary functions of time. The slow dynamics (4.12) then becomes

$$\frac{\mathrm{d}F(z)}{\mathrm{d}t} \text{ `='}\, \{F, H\} + U_{(l)}\{F, \phi_{(l)}\} + w_j\{F, \Phi_j\}, \tag{4.18}$$

where

$$\Phi_j \equiv e_{(l)j}\, \phi_{(l)}. \tag{4.19}$$

In the exact case considered by Dirac, the arbitrary functions $w_j(t)$ correspond to physically irrelevant choices of gauge. However, in the general approximation theory considered here, changes in $w_j(t)$ might conceivably cause physical differences, signalling a flaw in the initial choice of constraints. In practice, this must always be checked. With this single caveat, the general approximation theory is complete except for guidelines on the choice of constraints.

If $\{\phi_{(l)}, \phi_{(m)}\}$ is non-singular then the approximation (4.12)–(4.13) has a simple interpretation in the more abstract language of differential forms. Let the exact dynamics (3.1) be expressed

$$i_X\, \omega = \mathrm{d}H, \tag{4.20}$$

where X is the vector $\mathrm{d}z^i/\mathrm{d}t$, $i_X\, \omega$ is the contraction of X with the closed two-form ω corresponding to $\{\ ,\ \}$, and $\mathrm{d}H$ is the exterior derivative of the Hamiltonian H. Then the slow dynamics is equivalent to

$$i_X\, \omega|_M = \mathrm{d}H|_M, \tag{4.21}$$

where $\alpha|_M$ denotes the restriction of a differential form α to the slow manifold M. See, for example, Schutz (1980, p. 120). However, I much prefer the coordinate notation, which is anyway needed for calculations.

We conclude this section by demonstrating how Dirac's algorithm works in a familiar case. We again regard the shallow-water equations as exact, and consider the trial constraints

$$h(\boldsymbol{x}) = h_0 \text{ (constant)}. \tag{4.22}$$

Then (4.13) becomes

$$0 = \{h(\boldsymbol{x}), H\} + \iint \mathrm{d}\boldsymbol{x}'\, \mu_h(\boldsymbol{x}')\, \{h(\boldsymbol{x}), h(\boldsymbol{x}')\}. \tag{4.23}$$

By (3.7*b*), $\{h(\boldsymbol{x}), h(\boldsymbol{x}')\} = 0$. Thus μ_h is undetermined, and we have the consistency requirement

$$\{h, H\} = -\boldsymbol{\nabla}\cdot(\boldsymbol{u}h) = -h_0 \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{4.24}$$

We now start all over again with the combined constraints

$$h(\boldsymbol{x}) = h_0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{4.25}$$

The conditions (4.13) become

$$0 = \{h(\boldsymbol{x}), H\} + \iint \mathrm{d}\boldsymbol{x}'\mu_\Delta(\boldsymbol{x}')\{h(\boldsymbol{x}), \Delta(\boldsymbol{x}')\}, \tag{4.26}$$

and
$$0 = \{\Delta(\boldsymbol{x}), H\} + \iint \mathrm{d}\boldsymbol{x}'\mu_h(\boldsymbol{x}')\{\Delta(\boldsymbol{x}), h(\boldsymbol{x}')\} + \iint \mathrm{d}\boldsymbol{x}'\mu_\Delta(\boldsymbol{x}')\{\Delta(\boldsymbol{x}), \Delta(\boldsymbol{x}')\}, \tag{4.27}$$

where $\Delta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{u}$. By (3.7*b*)

$$\{h(\boldsymbol{x}), \Delta(\boldsymbol{x}')\} = -\nabla^2\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{4.28}$$

Thus (4.26) reduces to
$$\nabla^2\mu_\Delta = 0, \tag{4.29}$$

on (4.25). This implies $\mu_\Delta \equiv 0$ in the infinite geometry considered. Then (4.27) reduces to

$$0 = -\boldsymbol{\nabla}\cdot[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}] - \nabla^2\mu_h, \tag{4.30}$$

which determines μ_h. The slow dynamics (4.12) is

$$\begin{aligned}
\frac{\mathrm{d}F[\boldsymbol{u}]}{\mathrm{d}t} &= \{F, H\} + \iint \mathrm{d}\boldsymbol{x}'\mu_h(\boldsymbol{x}')\{F, h(\boldsymbol{x}')\} \\
&= \iint \mathrm{d}\boldsymbol{x}\frac{\delta F}{\delta u_i(\boldsymbol{x})}\left[\{u_i(\boldsymbol{x}), H\} + \iint \mathrm{d}\boldsymbol{x}'\mu_h(\boldsymbol{x}')\{u_i(\boldsymbol{x}), h(\boldsymbol{x}')\}\right] \\
&= \iint \mathrm{d}\boldsymbol{x}\frac{\delta F}{\delta u_i(\boldsymbol{x})}\left[-\boldsymbol{u}\cdot\boldsymbol{\nabla}u_i + \iint \mathrm{d}\boldsymbol{x}'\mu_h(\boldsymbol{x}')\frac{\partial}{\partial x_i}\delta(\boldsymbol{x}-\boldsymbol{x}')\right] \\
&= \iint \mathrm{d}\boldsymbol{x}\frac{\delta F}{\delta \boldsymbol{u}(\boldsymbol{x})}\cdot[-(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} - \boldsymbol{\nabla}\mu_h].
\end{aligned} \tag{4.31}$$

With $F = \boldsymbol{u}(\boldsymbol{x})$, (4.30) and (4.31) are equivalent to Euler's equations for a two-dimensional incompressible fluid. The coefficient μ_h turns out to be the pressure. This derivation by way of Dirac's algorithm applied to (4.22) seems very roundabout, but it nicely demonstrates how the algorithm automatically enlarges a set of trial constraints to produce a consistent slow dynamics.

5. Guidelines on the choice of constraints

The preceding section shows how to test and (if necessary and possible) how to augment a set of trial constraints to define a consistent slow manifold. With the slow manifold defined, the slow dynamics are uniquely determined by (4.12), (4.13). But how do we select these constraints in the first place? No general algorithm has been found, but the following summary of textbook results suggests that the general strategy followed by Salmon (1985) is a good one. Many choices of slow manifold will filter out high-frequency motions, but the best choice is that for which the resulting approximate dynamical equations take their simplest mathematical form.

We again consider an N-dimensional phase space with general coordinates z^i and exact dynamics determined by the Poisson bracket (3.4) and Hamiltonian $H(z)$. We now assume that J^{ij} is non-singular so that canonical coordinates exist. This is not a serious restriction, because the formulation of fluid mechanics in Lagrangian variables always has this property. Let ω_{ij} be the inverse of J^{ij},

$$\omega_{ij} J^{jk} = \delta_i^k \tag{5.1}$$

and let $F_1(z), F_2(z), \dots, F_N(z)$ be N independent phase functions. Then the Lagrange bracket of F_i and F_j is defined by

$$[F_i, F_j] \equiv \frac{\partial z^m}{\partial F_i} \omega_{mn} \frac{\partial z^n}{\partial F_j}. \tag{5.2}$$

Thus

$$\omega_{ij} = [z^i, z^j]. \tag{5.3}$$

It follows from (3.2) that

$$\frac{\partial \omega_{ij}}{\partial z^k} + \frac{\partial \omega_{jk}}{\partial z^i} + \frac{\partial \omega_{ki}}{\partial z^j} = 0. \tag{5.4}$$

Now let $\phi^\alpha(z), \alpha = 1$ to $N-M$, be $N-M$ consistent constraint functions defining an M-dimensional slow manifold. (We now write $\phi^\alpha(z)$ instead of $\phi_{(\alpha)}(z)$ to emphasize that the ϕ^α can serve as $N-M$ of the coordinates.) We assume that no arbitrary functions of time occur in the slow dynamics. Then $\{\phi^\alpha, \phi^\beta\}$ is non-singular, and the slow dynamics is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}_{\mathrm{D}}, \tag{5.5}$$

where

$$\{F, G\}_{\mathrm{D}} \equiv \{F, G\} - \{F, \phi^\alpha\}\{\phi^\alpha, \phi^\beta\}^{-1}\{\phi^\beta, G\}, \tag{5.6}$$

is the Dirac bracket.

Next let y^i be any M coordinates that cover the slow manifold. Then $(y^1, \dots, y^M, \phi^1, \dots, \phi^{N-M})$ are a set of N independent coordinates that cover the whole phase space. Let

$$\hat{F}(y) \equiv F(y, \phi = 0), \tag{5.7}$$

be the restriction of F to the slow manifold. Then

$$\{F, G\}_{\mathrm{D}} = \{\hat{F}, \hat{G}\}_{\mathrm{D}}, \tag{5.8}$$

and the slow dynamics can be written

$$\frac{\mathrm{d}\hat{F}}{\mathrm{d}t} = \{\hat{F}, \hat{H}\}_{\mathrm{D}}. \tag{5.9}$$

The crucial result is that

$$\{\hat{F}, \hat{G}\}_{\mathrm{D}} = \frac{\partial \hat{F}}{\partial y^i}[y^i, y^j]^{-1}\frac{\partial \hat{G}}{\partial y^j}. \tag{5.10}$$

Equation (5.10) follows from (5.3)–(5.6) (see, for example, Sudarshan & Mukunda 1983, pp. 120–122). Equation (5.4) further implies that

$$\hat{J}^{ij} \equiv [y^i, y^j]^{-1}, \tag{5.11}$$

obeys (3.2). Thus (5.9)–(5.10) satisfies the hypothesis of Darboux's theorem: The slow dynamics is a Hamiltonian dynamics on the slow manifold, and canonical coordinates exist. This shows what can be accomplished in principle. In practice, canonical coordinates might be hard to find, and the Hamiltonian $\hat{H}$ might have a complicated dependence on the canonical coordinates. However, we have the flexibility to adjust the ϕ^α defining the slow manifold, and to replace $\hat{H}$ by any reasonable approximation upon it. Salmon (1985) used both of these strategies.

As already explained in §4, Salmon (1985, §2) obtained the first of his semigeostrophic approximations by replacing

$$\iint \mathrm{d}\boldsymbol{a}[u_i(\boldsymbol{a}) - \tfrac{1}{2}f\epsilon_{ij}\,x_j(\boldsymbol{a})]\,\delta x_i(\boldsymbol{a}), \tag{5.12}$$

the analogue of $p_i\,\mathrm{d}q_i$ in the exact Lagrangian of (4.1), by

$$\iint \mathrm{d}\boldsymbol{a}[u_{\mathrm{G}i}(\boldsymbol{a}) - \tfrac{1}{2}f\epsilon_{ij}\,x_j(\boldsymbol{a})]\,\delta x_i(\boldsymbol{a}), \tag{5.13}$$

where $\boldsymbol{u}_{\mathrm{G}} \equiv \boldsymbol{k} \times \boldsymbol{\nabla}(gh/f)$ is the geostrophic velocity, a functional of $\boldsymbol{x}(\boldsymbol{a}, t)$. The equation for the slow manifold corresponding to (5.13) is thus

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{G}}. \tag{5.14}$$

Unfortunately, (5.13) does not have the form of canonical coordinates. Canonical coordinates *exist* for (5.13) by the foregoing general theory, but they are hopelessly difficult to calculate.

Salmon (1985, §3) overcame this difficulty by noting that (5.12) could be replaced by

$$\iint \mathrm{d}\boldsymbol{a}[-\tfrac{1}{2}f\epsilon_{ij}\,X_j(\boldsymbol{a})]\,\delta X_i(\boldsymbol{a}), \tag{5.15}$$

instead of (5.13), where

$$X_i \equiv x_i + \epsilon_{ij}\,u_{\mathrm{G}j}/f, \tag{5.16}$$

are Hoskins' 'geostrophic coordinates'. The expression (5.15) has the form of canonical coordinates, and it differs from (5.12) by the same size error as does (5.13).

Now either $x_i(\boldsymbol{a})$ or $X_i(\boldsymbol{a})$ cover the slow manifold. Substitution of (5.16) into (5.15) and straightforward but tedious calculations show that

$$(5.15) = \iint \mathrm{d}\boldsymbol{a}[u_{\mathrm{G}i}(\boldsymbol{a}) - \tfrac{1}{2}f\epsilon_{ij}\,x_j(\boldsymbol{a}) + (2f)^{-1}\,\epsilon_{ij}(\boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla})\,u_{\mathrm{G}j}]\,\delta x_i(\boldsymbol{a}). \tag{5.17}$$

Equating the coefficients of $\delta x_i(\boldsymbol{a})$ in (5.12) with those in (5.17), we discover the constraints

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{G}} + (2f)^{-1}(\boldsymbol{u}_{\mathrm{G}}\cdot\boldsymbol{\nabla})\,(\boldsymbol{u}_{\mathrm{G}} \times \boldsymbol{k}) \tag{5.18}$$

that define the slow manifold for the semigeostrophic equations of Salmon (1985, §3). The last term in (5.18) is small for nearly geostrophic flow.

In summary, the tremendous simplification of canonical coordinates has been achieved by changing the slow manifold, by a tiny amount, from (5.14) to (5.18). The constraints (5.18) are more complicated than (5.14), but the corresponding dynamical equations are much simpler. It is much better to have simple equations for the slow dynamics than it is to have simple equations for the slow manifold, because the latter are needed only for transforming results back into the original coordinates.

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