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Large-scale semigeostrophic equations for use in ocean circulation models

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Hamiltonian approximation methods yields approximate dynamical equations that apply to nearly geostrophic flow at scales larger than the internal Rossby deformation radius. These equations incorporate fluid inertia with the same order of accuracy as the semi-geostrophic equations, but are nearly as simple (in appropriate coordinates) as the equations obtained by completely omitting the inertia.

1. Introduction

Numerical models of the large-scale circulation in the Earth's ocean and atmosphere are usually based upon variants of the *primitive equations*, the equations of motion for a Boussinesq hydrostatic fluid. However, in addition to the slow geostrophically balanced motions of primary interest, solutions of the primitive equations contain relatively high-frequency inertia–gravity waves. In earlier days, these inertia–gravity waves severely hindered numerical computations, because the time-step could not exceed the time required for the fastest wave to travel between gridpoints. Sophisticated numerical methods (including semi-implicit methods) have largely overcome this difficulty, but the inertia–gravity waves remain troublesome, because relatively small errors in the computation of the geostrophic motions can excite unrealistically large inertia–gravity waves. Quite apart from the issues of efficiency and accuracy, the presence of inertia–gravity waves and small-scale turbulence greatly complicates the interpretation of solutions.

Balanced equations are approximations to the primitive equations that filter out the inertia–gravity waves but retain the geostrophically balanced motions. The simplest such equations are the *planetary geostrophic equations*, which completely omit the fluid inertia. More typical balanced equations take account of the inertia in the geostrophic motions; the *quasi-geostrophic equations* and the *semi-geostrophic equations* are among the most useful of such approximations. However, the semi-geostrophic equations are relatively difficult to solve (in their general three-dimensional form), and the quasi-geostrophic equations are artificially restricted to flows in which the isopycnal surfaces are nearly flat.

In a series of papers, Salmon (1983, 1985, 1988*a*) proposed a strategy for constructing new balanced equations, in which the approximations corresponding to balance are imposed as constraints on the Lagrangian for the primitive equations. This strategy has two important advantages. First, the conservation laws for energy and potential vorticity survive if the approximations respect the corresponding symmetry properties. Second, transformations to new dependent and independent variables, in which the approximate dynamics takes its simplest form, automatically suggest themselves.

In this paper we extend an idea proposed by Salmon (1985, §4) to obtain approximate equations for nearly geostrophic flow at horizontal lengthscales larger than the *internal* Rossby deformation radius Nh/f, where N is the Brunt–Väisälä frequency, h is the fluid depth, and f is the Coriolis parameter. These *large-scale semi-geostrophic equations* (LSGE) incorporate inertia at the order of the semi-geostrophic equations, but are nearly as simple and easy to solve as the (inertia-less) planetary geostrophic equations.

This paper is self-contained, but it should be considered a sequel to Salmon (1985). We begin, in §2, with a brief overview of the primitive, planetary geostrophic and semi-geostrophic approximations. Section 3 reviews the Hamiltonian approximation theory developed in the earlier papers. In §§4 and 5, we derive the LSGE and demonstrate their close mathematical resemblance to the planetary geostrophic equations.

Because the planetary geostrophic equations conserve a very simple form of potential vorticity, the assumption that surfaces of constant buoyancy and potential vorticity coincide leads to a useful exact reduction to a coupled pair of equations in *two* space dimensions. This reduction formed the basis for a simple numerical ocean circulation model developed by Salmon (1994). In §5 we show that the LSGE conserve a form of potential vorticity that is nearly as simple (in appropriate coordinates) as that of the planetary geostrophic equations. Hence, Salmon's (1994) model can be modified to include the inertia without sacrificing its other advantages.

The present paper illustrates the enormous advantages of Hamiltonian approximation theory more convincingly than any of my earlier papers. Although the specific results are primarily interesting to oceanographers and meteorologists, the general strategy and philosophy exemplified here ought to be useful in many other applications.

2. The primitive equations and the planetary geostrophic equations

Consider an inhomogeneous rotating fluid with a free surface at $z = \eta(x, y, t)$ and a rigid lower boundary at $z = -h(x, y)$. The *primitive equations* of motion (expressing the Boussinesq and hydrostatic approximations) are

$$\epsilon\frac{\mathrm{D}u}{\mathrm{D}t}-fv = -\frac{\partial\phi}{\partial x}, \quad \epsilon\frac{\mathrm{D}v}{\mathrm{D}t}+fu = -\frac{\partial\phi}{\partial y}, \tag{2.1a}$$

$$0 = -\frac{\partial\phi}{\partial z}-g+\theta, \tag{2.1b}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{2.1c}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{2.1d}$$

with boundary conditions

$$\phi = 0 \quad \text{and} \quad w = \frac{\mathrm{D}\eta}{\mathrm{D}t} \quad \text{at} \quad z = \eta, \tag{2.2}$$

and

$$w = -\frac{\mathrm{D}h}{\mathrm{D}t} \quad \text{at} \quad z = -h(x, y). \tag{2.3}$$

Here, (u, v, w) is the fluid velocity in the direction of Cartesian coordinates (x, y, z) with z pointing up; $f(x, y)$ is the Coriolis parameter (that is, twice the local vertical component of the rotation vector); ϕ is the pressure; g is the gravity constant; θ is the buoyancy (which we shall call temperature); and $\mathrm{D}/\mathrm{D}t$ is the usual substantial derivative. Frequently, one assumes that $f = f_0 + \beta y$, where f_0 and β are constants, and y is northward distance, but it is more illuminating to allow an arbitrary dependence of f on the horizontal location (x, y). The formal parameter $\epsilon \equiv 1$ in (2.1) reminds us that we are interested in flows in which the relative accelerations are small compared to the Coriolis force, that is, in which the Rossby number is small.

The primitive equations (2.1)–(2.3) conserve the energy

$$E = \iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\{\tfrac{1}{2}\epsilon(u^2+v^2)+(g-\theta)\,z\}, \tag{2.4}$$

where the integration runs over the whole fluid, and the potential vorticity,

$$Q = \left(f+\epsilon\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right)\frac{\partial \theta}{\partial z}+\epsilon\frac{\partial u}{\partial z}\frac{\partial \theta}{\partial y}-\epsilon\frac{\partial v}{\partial z}\frac{\partial \theta}{\partial x}, \tag{2.5}$$

on fluid particles, $\mathrm{D}Q/\mathrm{D}t = 0$.

We are interested in cases in which the flow is nearly geostrophic, that is, in which the ϵ-terms in (2.1) are small, and we seek approximate dynamical equations that are simpler than (2.1). In the most drastic approximation, we simply set $\epsilon = 0$ in (2.1). The resulting *planetary geostrophic equations*,

$$-fv = -\frac{\partial \phi}{\partial x}, \quad fu = -\frac{\partial \phi}{\partial y}, \tag{2.6a, b}$$

$$0 = -\frac{\partial \phi}{\partial z} - g + \theta, \tag{2.6c}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{2.6d}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{2.6e}$$

completely omit the inertia. Although the planetary geostrophic equations have been rather widely used in oceanography (see, for example, Salmon 1994), their complete neglect of inertia is probably too severe to explain many realistic features of the large-scale ocean circulation.

The search for approximate equations for nearly geostrophic flow with accuracy between (2.1) and (2.6) has been longstanding; see, for example, McWilliams & Gent (1980) and Allen, Barth & Newberger (1990). Of the many equations proposed, the *semi-geostrophic equations* (Hoskins 1975; Cullen & Purser 1989) seem to be especially promising. The semi-geostrophic equations filter out the high-frequency inertia–gravity waves present in solutions of (2.1), while exactly conserving logical low-Rossby-number approximations to the energy and the potential vorticity on particles. Moreover, the semi-geostrophic equations have a Hamiltonian structure (Salmon 1985, 1988*a*; Purser 1993), which at least partly accounts for their remarkable conservation laws and beautiful transformation properties. Finally, unlike the well-known *quasi-geostrophic equations*, the semi-geostrophic equations do not demand that the fluid depth and the vertical separation between isothermal surfaces be nearly uniform.

In the case of constant Coriolis parameter f, the semi-geostrophic equations are equivalent to the equations obtained by replacing

$$\epsilon \frac{\mathrm{D}}{\mathrm{D}t}(u, v) \rightarrow \epsilon \frac{\mathrm{D}}{\mathrm{D}t}(u_G, v_G) \tag{2.7}$$

in (2.1), where (u_G, v_G) is the geostrophic velocity, defined by (2.9). In the generalization to non-constant Coriolis parameter $f(x,y)$ proposed by Salmon (1985), the semi-geostrophic equations take a relatively complicated form in ordinary physical coordinates (x, y, z). In both cases, however, the semi-geostrophic equations take their simplest form in so-called *geostrophic coordinates* (x_s, y_s, z_s) defined (implicitly) by

$$x_s \equiv x + \epsilon \frac{v_G}{f(x_s, y_s)}, \quad y_s \equiv y - \epsilon \frac{u_G}{f(x_s, y_s)}, \quad z_s \equiv z, \tag{2.8a–c}$$

where

$$u_G \equiv -\frac{1}{f(x_s, y_s)} \frac{\partial \pi}{\partial y}, \quad v_G \equiv \frac{1}{f(x_s, y_s)} \frac{\partial \pi}{\partial x}, \tag{2.9a, b}$$

and

$$\pi \equiv g(\eta - z) - \int_z^\eta \theta \, \mathrm{d}z' \tag{2.10}$$

is the hydrostatic pressure. The geostrophic coordinates turn out to be *canonical coordinates* in the Hamiltonian formulation of the theory.

Despite their many favourable properties, the semi-geostrophic equations are significantly harder to solve than (say) the quasi-geostrophic equations. A central difficulty is that solutions of the semi-geostrophic equations can (and normally will) evolve into states for which the equations are ill-posed, that is, in which the semi-geostrophic equation for the pressure-tendency becomes non-elliptic. The conditions for non-ellipticity turn out to be the same as the conditions for symmetric and static instability of the flow (Shutts & Cullen 1987), and the ill-posedness thus corresponds to the development of localized regions of static and symmetric instability that the approximate dynamics evidently cannot relieve. Well-posedness can be maintained by adding eddy-friction and diffusion terms (perhaps adjusted to be largest in the regions of localized instability). However, Cullen & Purser (1984, 1989; see also Shutts, Cullen & Chynoweth 1988 and Cullen, Norbury & Purser 1991) have developed a substantial generalization of semi-geostrophic theory in which the instabilities leading to ill-posedness are automatically relieved without the addition of a large *ad hoc* eddy diffusion or the loss of conservation laws. In this so-called *geometric theory*, the elliptic equation for the pressure tendency is replaced by a (nonlinear) Monge–Ampère equation for a scalar whose curvature, in geostrophic coordinates, is the (inverse) potential vorticity. The solution of the Monge–Ampère equation is subject to a convexity condition (equivalent to static and inertial stability) that guarantees its uniqueness but may imply that the fluid particles undergo discontinuous rearrangements. However, despite the very elegant generalization of Cullen, Purser and collaborators, the semi-geostrophic equations remain difficult to solve. In fact, there seem to be few three-dimensional numerical solutions of the semi-geostrophic equations, especially in domains, like the ocean, with complicated boundary shapes.

In this paper, we combine the assumption of nearly geostrophic motion with the additional assumption that the flow of interest has horizontal lengthscales larger than the internal deformation radius, to obtain approximate dynamical equations that are

Hamiltonian, conservative, and incorporate the effects of inertia at the same order of approximation as do the semi-geostrophic equations. However (because they take advantage of the restriction to scales larger than the internal deformation radius) the new equations are scarcely more complicated than the planetary geostrophic equations (2.6). We call these new equations (which are a generalization of those proposed by Salmon (1985, §4)) the *large-scale semi-geostrophic equations* (LSGE). The LSGE are nearly as simple to solve as (2.6), but they do not apply accurately to lengthscales smaller than the internal deformation radius. However, numerical ocean-circulation models seldom resolve lengthscales smaller than the internal deformation radius, which is about 40 km at mid-latitude.

Like the planetary geostrophic equations, the LSGE admit a very useful, exact reduction to a pair of coupled equations in two space dimensions. In the case of the planetary geostrophic equations (2.6), this reduction is possible because the conservation laws for temperature and potential vorticity,

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(f\theta_z) = 0, \tag{2.11}$$

imply that the ansatz

$$f\theta_z = G(\theta), \tag{2.12}$$

where $G(\ )$ is an arbitrary function, is consistent with (2.6) and the boundary conditions (2.2) and (2.3). However, (2.12) integrates immediately to

$$\theta = \Theta\left(\frac{z}{f} + S(x,y,t)\right), \tag{2.13}$$

where $\Theta(\ )$ is another arbitrary function, related to G, and $S(x,y,t)$ is a function of integration, independent of z, to be determined by substituting (2.13) back into (2.6) and (2.2)–(2.3). The resulting two-dimensional equations are a very handy generalization of the two-layer analogue of (2.6), to which they reduce in the case in which the arbitrary function $\Theta(\ )$ is chosen to be a step (i.e. Heaviside) function. For many further details, see Salmon (1994).

Now, obviously, no such strategy could apply to the primitive equations (2.1); the analogue of (2.12), namely $Q = G(\theta)$ with Q given by (2.5), is a partial differential equation in u, v and θ that cannot be directly integrated, as could (2.12) to (2.13). However, the LSGE analogue of (2.12) turns out to be nearly as simple as (2.12), and can be directly integrated to yield a simple generalization of the ansatz (2.13), provided that all the calculations are performed in special coordinates that are the natural coordinates in which to formulate the theory. These special coordinates bear the same relation to the LSGE as do (2.8) to the semi-geostrophic equations. Thus, the LSGE have virtually all the useful mathematical properties of (2.6), but correctly incorporate inertia at the same order of accuracy as the semi-geostrophic equations. Again, their primary disadvantage is their limitation to lengthscales larger than the internal deformation radius.

Shutts (1989) used Hamilton's principle to derive equations for nearly geostrophic flow in which the fluid velocity in the direction of the Earth's rotation vector is assumed to be small. Since this component of velocity includes a large fraction of the northward velocity, Shutt's equations apply to very large-scale atmospheric flow (in which the velocity is predominantly zonal) but are not as useful for the ocean (in which some of the strongest geostrophic currents are northward-flowing boundary currents like the Gulf Stream).

3. Hamiltonian approximation theory

We derive the large-scale semi-geostriphic equations by making approximations based upon our two fundamental assumptions directly to the Lagrangian corresponding to the primitive equations (2.1). This strategy has two important advantages. First, conservation laws survive if the approximations respect the corresponding symmetry properties. Secondly, transformations to new dependent and independent variables, in which the approximate dynamics takes it simplest form, automatically suggest themselves.

Let $\boldsymbol{x}(\boldsymbol{a},\tau) = (x(a,b,c,\tau), y(a,b,c,\tau), z(a,b,c,\tau))$ be the location of the fluid particle identified by the labelling coordinates (a,b,c) at time τ. The labelling coordinates move with the flow. Thus $\partial/\partial\tau$ is the time-derivative following a fluid particle, and $\partial/\partial\tau \equiv \mathrm{D}/\mathrm{D}t$. We assign the labelling coordinates so that $\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \mathrm{d}(mass)$. Then the Lagrangian corresponding to the primitive equations is

$$L = \iiint \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\left\{(\epsilon u - R(x,y))\frac{\partial x}{\partial\tau} + (\epsilon v + P(x,y))\frac{\partial y}{\partial\tau}\right\} - H, \tag{3.1}$$

where

$$H = \iiint \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\left\{\tfrac{1}{2}\epsilon(u^2+v^2) + (g-\theta(a,b,c))\,z - \phi\left[\frac{\partial(x,y,z)}{\partial(a,b,c)} - \alpha_0\right]\right\} \tag{3.2}$$

is the Hamiltonian. In (3.1) and (3.2) the integrations run over the full mass of fluid. The prescribed functions $R(x,y)$ and $P(x,y)$ are any two functions satisfying $\partial R/\partial y + \partial P/\partial x = f(x,y)$. The temperature $\theta(a,b,c)$ is a prescribed function (depending on initial conditions) of the particle identity only (and thus $\partial\theta/\partial\tau = 0$). Hamilton's principle states that $\delta\int L\,\mathrm{d}\tau = 0$, for arbitrary independent variations $\delta\boldsymbol{x}(\boldsymbol{a},\tau)$, $\delta u(\boldsymbol{a},\tau)$, $\delta v(\boldsymbol{a},\tau)$, $\delta\phi(\boldsymbol{a},\tau)$ in the particle locations and velocities, and in the Lagrange multiplier ϕ for the incompressibility constraint,

$$\delta\phi:\quad \frac{\partial(x,y,z)}{\partial(a,b,c)} = \alpha_0, \tag{3.3}$$

where α_0 is the constant specific volume of the fluid. The τ-derivative of (3.3) is (2.1 *c*). Hamilton's principle yields (2.1 *a, b*) and the surface boundary condition (2.2). We incorporate the bottom boundary condition by constraining the fluid particle trajectories and their variations to be tangent to the rigid bottom at $z = -h(z,y)$. The Lagrangian (3.1) differs from the corresponding Lagrangian for a general (Boussinesq) fluid only in that the terms containing the vertical velocity w have been dropped; this neglect corresponds to the hydrostatic approximation. For further background, see Salmon (1988 *b*).

The conservation of energy (2.4) and potential vorticity (2.5) correspond, respectively, to the symmetry properties that the Hamiltonian (3.2) is invariant with respect to time-translations, and to relabellings of the fluid particles that do not affect $\theta(a,b,c)$ or the Jacobian in (3.3). Because of its importance in what follows, we will examine the particle-relabelling symmetry more closely. First, realize that Hamilton's principle requires that the action be stationary with respect to variations in the time-dependent mapping $\boldsymbol{x}(\boldsymbol{a},\tau;\mu)$ from label-space with coordinates (a,b,c) to physical-space with coordinates (x,y,z). Here, μ is a parameter that controls the variation, i.e.

$$\delta\boldsymbol{x}(a,b,c,\tau) \equiv \frac{\partial\boldsymbol{x}}{\partial\mu}(a,b,c,\tau;0), \tag{3.4}$$

and $\mu = 0$ corresponds to the unvaried particle trajectory. Now, Hamilton's principle is obviously equivalent to the requirement that the action be stationary with respect to variations in the inverse mapping $\boldsymbol{a}(\boldsymbol{x}, \tau; \mu)$ from (x, y, z)-space to (a, b, c)-space. By writing

$$x = x(a(x, y, z, t; \mu), b(x, y, z, t; \mu), c(x, y, z, t; \mu), t; \mu), \tag{3.5}$$

regarding each side of (3.5) as a function of (x, y, z, t, μ), and taking $\partial/\partial\mu$, we obtain the equation

$$\frac{\partial x}{\partial \mu} = -\frac{\partial x}{\partial a}\frac{\partial a}{\partial \mu} - \frac{\partial x}{\partial b}\frac{\partial b}{\partial \mu} - \frac{\partial x}{\partial c}\frac{\partial c}{\partial \mu} \tag{3.6}$$

relating variations in the mapping and its inverse. That is,

$$\delta x = -\frac{\partial x}{\partial a}\delta a - \frac{\partial x}{\partial b}\delta b - \frac{\partial x}{\partial c}\delta c. \tag{3.7}$$

Now, since $\theta(a, b, c)$ remains constant on each fluid particle, we can take the temperature itself as one of the particle labels. Suppose then that $\theta \equiv c$. Then, particle-label variations satisfying

$$\delta \frac{\partial(a, b, c)}{\partial(x, y, z)} = 0 \quad \text{and} \quad \delta c = 0 \tag{3.8}$$

leave the Hamiltonian (3.2) unchanged. For variations satisfying (3.8), Hamilton's principle thus implies that

$$\begin{aligned} 0 &= \delta \int \mathrm{d}t \iiint \mathrm{d}\boldsymbol{x} \frac{\partial(\boldsymbol{a})}{\partial(\boldsymbol{x})} \left\{ (\epsilon u - R(x, y)) \frac{\partial x}{\partial \tau} + (\epsilon v + P(x, y)) \frac{\partial y}{\partial \tau} \right\} \\ &= \int \mathrm{d}t \iiint \mathrm{d}\boldsymbol{x} \frac{\partial(\boldsymbol{a})}{\partial(\boldsymbol{x})} \left\{ (\epsilon u - R(x, y))\, \delta \frac{\partial x}{\partial \tau} + (\epsilon v + P(x, y))\, \delta \frac{\partial y}{\partial \tau} \right\}. \end{aligned} \tag{3.9}$$

By steps similar to those leading to (3.6), we find that

$$\delta \frac{\partial x}{\partial \tau} = -\frac{\partial x}{\partial a}\frac{\partial}{\partial \tau}\delta a - \frac{\partial x}{\partial b}\frac{\partial}{\partial \tau}\delta b - \frac{\partial x}{\partial c}\frac{\partial}{\partial \tau}\delta c. \tag{3.10}$$

But, by (3.8), $\delta c = 0$ and hence

$$\delta a = -\frac{\partial}{\partial b}\delta\psi, \quad \delta b = +\frac{\partial}{\partial a}\delta\psi, \tag{3.11}$$

for some (arbitrary) $\delta\psi(a, b, c, \tau)$. After integrations by parts, (3.9) thus becomes

$$0 = \int \mathrm{d}\tau \iiint \mathrm{d}\boldsymbol{a}\, \delta\psi(\boldsymbol{a}, \tau) \frac{\partial}{\partial \tau} \left\{ \frac{\partial(x, \epsilon u - R, c)}{\partial(a, b, c)} + \frac{\partial(y, \epsilon v + P, c)}{\partial(a, b, c)} \right\}. \tag{3.12}$$

Then since $\delta\psi$ is arbitrary, (3.12) implies that $\partial Q/\partial\tau = 0$ where

$$\begin{aligned} Q &= \frac{\partial(x, \epsilon u - R, c)}{\partial(a, b, c)} + \frac{\partial(y, \epsilon v + P, c)}{\partial(a, b, c)} = \frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{a})} \left[\frac{\partial(x, \epsilon u - R, c)}{\partial(x, y, z)} + \frac{\partial(y, \epsilon v + P, c)}{\partial(x, y, z)} \right] \\ &= \frac{\partial(x, \epsilon u - R, \theta)}{\partial(x, y, z)} + \frac{\partial(y, \epsilon v + P, \theta)}{\partial(x, y, z)} \end{aligned} \tag{3.13}$$

is equivalent to (2.5).

All the dynamical approximations considered in this paper correspond to replacements

$$u(\boldsymbol{a},\tau) \leftarrow u[x(\boldsymbol{a},\tau), y(\boldsymbol{a},\tau), z(\boldsymbol{a},\tau)] \quad \text{and} \quad v(\boldsymbol{a},\tau) \leftarrow v[x(\boldsymbol{a},\tau), y(\boldsymbol{a},\tau), z(\boldsymbol{a},\tau)], \tag{3.14}$$

in the Lagrangian (3.1), of the momentum variables $u(\boldsymbol{a},\tau)$ and $v(\boldsymbol{a},\tau)$ by prescribed functionals $u[\boldsymbol{x}(\boldsymbol{a},\tau)]$ and $v[\boldsymbol{x}(\boldsymbol{a},\tau)]$ of the location variables $\boldsymbol{x}(\boldsymbol{a},\tau)$. The replacements (3.14) are projective in the sense that the resulting Lagrangian depends only on the particle locations $\boldsymbol{x}(\boldsymbol{a},\tau)$ and the Lagrange-multiplier field $\phi(\boldsymbol{a},\tau)$, that is, on two fewer fields than (3.1) and (3.2).

The planetary geostrophic equations (2.6) result from the replacements

$$u \leftarrow 0 \quad \text{and} \quad v \leftarrow 0 \tag{3.15}$$

in (3.1) and (3.2). The resulting Lagrangian has an error of $O(\epsilon)$.

The semi-geostrophic equations result from the more accurate replacements

$$u \leftarrow u_G[x,y,z] \quad \text{and} \quad v \leftarrow v_G[x,y,z], \tag{3.16}$$

where (u_G, v_G) is the geostrophic velocity defined by (2.9) and (2.10). The resulting Lagrangian, which has an $O(\epsilon^2)$ error, is then renormalized, to make it resemble the simpler Lagrangian for the planetary geostrophic equations. This renormalization step is based upon the observation that

$$\iiint \mathrm{d}\boldsymbol{a}\{(\epsilon u - R(x,y))\,\delta x + (\epsilon v + P(x,y))\,\delta y\} = \iiint \mathrm{d}\boldsymbol{a}\{-R(x_s,y_s)\,\delta x_s + P(x_s,y_s)\,\delta y_s\}, \tag{3.17}$$

to within an error of $O(\epsilon^2)$ and an irrelevant total functional differential, where $(\delta x, \delta y)$ are arbitrary, and $(\delta x_s, \delta y_s)$ are the corresponding variations of the geostrophic coordinates defined by (2.8). The simplification (3.17) is of course the motivation for the definition (2.8), which amounts to a transformation to canonical variables. For further explanation, refer to Salmon (1985).

The semi-geostrophic equations are equivalent to the statement that $\delta \int L_{SG}\,\mathrm{d}\tau = 0$, where

$$L_{SG} = \iiint \mathrm{d}\boldsymbol{a}\left\{-R(x_s,y_s)\frac{\partial x_s}{\partial \tau} + P(x_s,y_s)\frac{\partial y_s}{\partial \tau}\right\} - H_{SG}, \tag{3.18}$$

and

$$H_{SG} = \iiint \mathrm{d}\boldsymbol{a}\left\{\tfrac{1}{2}\epsilon(u_G^2+v_G^2) + (g-\theta(a,b,c))\,z - \phi\left[\frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{a})} - \alpha_0\right]\right\}, \tag{3.19}$$

for arbitrary variations $\delta\boldsymbol{x}_s(\boldsymbol{a},\tau)$ and $\delta\phi(\boldsymbol{a},\tau)$ in the geostrophic coordinates $\boldsymbol{x}_s \equiv (x_s, y_s, z_s)$ and in the Lagrange multiplier ϕ. (The derivation is a straightforward generalization of that given by Salmon (1985, §3 and Appendix B).) However, the variations of H_{SG} are somewhat awkward to take, and the resulting semi-geostrophic equations somewhat complicated, because the Hamiltonian (3.19) does not take a particularly simple form in the new coordinates $\boldsymbol{x}_s$. That is, the new coordinates simplify the first part of the Lagrangian (3.18) (the part corresponding to the Poisson bracket, in the usual geometrical view of the subject), but the Hamiltonian (3.19) takes a much simpler form in the old coordinates $\boldsymbol{x}$. We shall see that it is possible to rewrite the Hamiltonian as a simple functional of the new coordinates, but only after an additional restriction.

To expose the essential difficulty, we temporarily assume that the rigid bottom at $z = -h$ is flat, that is, that $h = h_0$ is a constant. The constraint

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = \alpha_0 \tag{3.20}$$

in (3.19) takes a complicated form in the geostrophic coordinates (2.8). However, redefining the new vertical coordinate to be

$$z_s = z - \epsilon \int_{-h_0}^{z_s} \left[\frac{\partial}{\partial x_s}\left(\frac{v_G}{f_s}\right) - \frac{\partial}{\partial y_s}\left(\frac{u_G}{f_s}\right)\right] \mathrm{d}z_s' \tag{3.21}$$

instead of (2.8*c*), we find that (3.20) is equivalent to

$$\frac{\partial(x_s, y_s, z_s)}{\partial(a, b, c)} = \alpha_0, \tag{3.22}$$

to within the same $O(\epsilon^2)$ error already present in (3.18) and (3.19). Here, $f_s \equiv f(x_s, y_s)$ and the lower integration limit in (3.21) was chosen to make the equation for the rigid bottom take the simple form $z_s = -h$ in the new coordinates.

We now turn to the (u_G, v_G) terms in (3.19). These would take a simple form in the new coordinates if only we could accurately replace

$$\frac{\partial \pi}{\partial x} \leftarrow \frac{\partial \pi_s}{\partial x_s}, \quad \frac{\partial \pi}{\partial y} \leftarrow \frac{\partial \pi_s}{\partial y_s} \tag{3.23}$$

in the definitions (2.9), where

$$\pi_s \equiv g(\eta_s - z_s) - \int_{-h}^{\eta_s} \theta \, \mathrm{d}z_s', \tag{3.24}$$

and $z_s = \eta_s$ is the location of the free surface in the new coordinates. Unfortunately, the first term in (3.24) presents serious difficulties, because

$$\frac{g}{f_s}\frac{\partial \eta_s}{\partial x_s} \approx \frac{g}{f}\frac{\partial \eta_s}{\partial x} \approx \frac{g}{f}\frac{\partial}{\partial x}\left[\eta + \epsilon \int_{-h_0}^{\eta} \left[\left(\frac{v_G}{f}\right)_x - \left(\frac{u_G}{f}\right)_y\right] \mathrm{d}z'\right], \tag{3.25}$$

and the last two terms in (3.25) are of size U and ghU/f^2L^2 respectively, where U is the scale for the horizontal velocity, and L is the horizontal lengthscale of the flow. Thus the replacement (3.23) requires that L be much larger than the *external deformation radius* $\lambda_{ext} \equiv (gh)^{\frac{1}{2}}/f$. By similar steps, one can show that the last term in (3.24) is accurate if L is much larger than the internal deformation radius $\lambda_{int} \equiv Nh/f$, where N (the Brunt–Väisälä frequency) is a typical value of $\partial\theta/\partial z$.

The largest numerical ocean-circulation models resolve lengthscales as small as 20 km. Since the external deformation radius is about 2000 km, the requirement $L \gg \lambda_{ext}$ is much too severe. On the other hand, the internal deformation radius is only about 40 km. Thus $L \gg \lambda_{int}$ is a tolerable constraint. In the following section we show how it is possible to simplify the approximate Hamiltonian by using only the latter assumption. The trick is to divide the velocity into a vertically averaged part and a remainder, and to constrain the velocity by assuming that this remainder is determined by the temperature field, through the thermal wind relations. We treat the vertically averaged flow in much the same way as if we were making no approximations at all, and filter out the external-mode inertia–gravity waves with the rigid-lid approximation. Since the rigid-lid approximation corresponds to the assumption that $L \ll \lambda_{ext}$, our final equations, the large-scale semi-geostrophic equations, are accurate if the flow is nearly geostrophic, and if $\lambda_{int} \ll L \ll \lambda_{ext}$.

4. Derivation of the large-scale semi-geostrophic equations

Consider, then, a fluid bounded by a rigid lid at $z = 0$ and a rigid bottom at $z = -h(x,y)$. The Lagrangian for the primitive equations is (3.1) and (3.2), but the g-term is now irrelevant and can be dropped. At both the rigid lid and the rigid bottom, we require the particle trajectories and their variations to be tangent to the boundary.

In the primitive-equation Lagrangian, we now replace

$$u(\boldsymbol{a},\tau) \leftarrow \bar{u}+\hat{u}, \quad v(\boldsymbol{a},\tau) \leftarrow \bar{v}+\hat{v}, \tag{4.1}$$

where $(\bar{u},\bar{v})$ is the vertically averaged horizontal velocity, and $(\hat{u},\hat{v})$ the departure therefrom. Both are functionals of the fluid-particle locations, as defined below. We define the new coordinates

$$x_s(\boldsymbol{a},\tau) = x(\boldsymbol{a},\tau)+\epsilon\frac{\hat{v}}{f_s}, \tag{4.2a}$$

$$y_s(\boldsymbol{a},\tau) = y(\boldsymbol{a},\tau)-\epsilon\frac{\hat{u}}{f_s}, \tag{4.2b}$$

$$z_s(\boldsymbol{a},\tau) = z(\boldsymbol{a},\tau)+\epsilon\int_{z_s}^{0}\left[\frac{\partial}{\partial x_s}\left(\frac{\hat{v}}{f_s}\right)-\frac{\partial}{\partial y_s}\left(\frac{\hat{u}}{f_s}\right)\right]dz_s', \tag{4.2c}$$

where

$$(\bar{u},\bar{v}) \equiv \frac{1}{h_s}\int_{-h_s}^{0}\left(\frac{\partial x_s}{\partial \tau},\frac{\partial y_s}{\partial \tau}\right)dz_s', \tag{4.3}$$

and $(\hat{u},\hat{v})$ is defined by

$$\frac{\partial}{\partial z_s}(\hat{u},\hat{v}) = \frac{1}{f_s}\left(-\frac{\partial\theta}{\partial y_s},\frac{\partial\theta}{\partial x_s}\right) \quad \text{and} \quad \int_{-h_s}^{0}(\hat{u},\hat{v})\,dz_s = 0. \tag{4.4}$$

Here,

$$f_s \equiv f(x_s,y_s) \quad \text{and} \quad h_s \equiv h(x_s,y_s). \tag{4.5}$$

Thus $(\bar{u},\bar{v})$ is the vertically averaged velocity in (x_s,y_s,z_s) coordinates and, similarly, $(\hat{u},\hat{v})$ obeys the thermal wind equations in these same new coordinates. Since all the terms containing the horizontal velocity in the Lagrangian are already of $O(\epsilon)$, the definitions (4.2)–(4.4) are consistent with an overall accuracy of $O(\epsilon^2)$. These definitions will allow us to express the approximate Lagrangian neatly in terms of the new coordinates, and this leads to final equations of great simplicity.

Once again,

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\partial(x_s,y_s,z_s)}{\partial(a,b,c)}+O(\epsilon^2). \tag{4.6}$$

Moreover, by the definition (4.2c), the equation for the rigid upper lid, $z = 0$, transforms exactly to $z_s = 0$. Similarly, the equation for the rigid bottom, $z = -h(x,y)$, transforms to $z_s = -h(x_s,y_s)$, to within the required $O(\epsilon^2)$ accuracy of the approximation. To see this, note that $z = -h(x,y)$ is equivalent to

$$\begin{aligned} z_s &= -h(x,y)+\epsilon\int_{-h_s}^{0}\left[\frac{\partial}{\partial x_s}\left(\frac{\hat{v}}{f_s}\right)-\frac{\partial}{\partial y_s}\left(\frac{\hat{u}}{f_s}\right)\right]dz_s'+O(\epsilon^2) \\ &= -h(x_s,y_s)-\nabla h(x_s,y_s)\cdot(x-x_s,y-y_s)+\dots \\ &\quad +\epsilon\frac{\partial}{\partial x_s}\int_{-h_s}^{0}\frac{\hat{v}}{f_s}dz_s-\epsilon\frac{\partial}{\partial y_s}\int_{-h_s}^{0}\frac{\hat{u}}{f_s}dz_s-\epsilon\nabla h(x_s,y_s)\cdot\left(\frac{\hat{v}}{f_s},-\frac{\hat{u}}{f_s}\right) \\ &= -h(x_s,y_s)+O(\epsilon^2). \end{aligned} \tag{4.7}$$

Thus, to consistent order, the rigid-boundary conditions take the same simple form in the new coordinates as they did in the old coordinates. Note that the final step in (4.7) depends critically on the fact that the vertical integral of $(\hat{u}, \hat{v})$ vanishes.

The approximate Lagrangian is

$$L = \iiint \mathrm{d}\boldsymbol{a} \left\{ (\epsilon\bar{u} + \epsilon\hat{u} - R(x,y)) \frac{\partial x}{\partial \tau} + (\epsilon\bar{v} + \epsilon\hat{v} + P(x,y)) \frac{\partial y}{\partial \tau} \right\} - H, \tag{4.8}$$

where

$$H = \iiint \mathrm{d}\boldsymbol{a} \left\{ \tfrac{1}{2}\epsilon(\bar{u}^2 + \bar{v}^2) + \tfrac{1}{2}\epsilon(\hat{u}^2 + \hat{v}^2) - \theta z - \phi \left[\frac{\partial(\boldsymbol{x}_s)}{\partial(\boldsymbol{a})} - \alpha_0 \right] \right\}. \tag{4.9}$$

However, with the same justification as in §3, we can replace (4.8) by

$$L = \iiint \mathrm{d}\boldsymbol{a} \left\{ (\epsilon\bar{u} - R(x_s, y_s)) \frac{\partial x_s}{\partial \tau} + (\epsilon\bar{v} + P(x_s, y_s)) \frac{\partial y_s}{\partial \tau} \right\} - H. \tag{4.10}$$

We note that

$$\iiint \mathrm{d}\boldsymbol{a}\, \bar{u} \frac{\partial x_s}{\partial \tau} = \iint \mathrm{d}x_s\, \mathrm{d}y_s \int_{-h_s}^{0} \mathrm{d}z_s\, \bar{u} \frac{\partial x_s}{\partial \tau} = \iiint \mathrm{d}\boldsymbol{x}_s\, \bar{u}^2 = \iiint \mathrm{d}\boldsymbol{a}\, \bar{u}^2. \tag{4.11}$$

To further simplify H, we use (4.2c) to write

$$\iiint \mathrm{d}\boldsymbol{a}\{-z\theta\} = -\iiint \mathrm{d}\boldsymbol{a}[z_s + \epsilon \boldsymbol{\nabla}_s \cdot \boldsymbol{A}]\, \theta, \tag{4.12}$$

where $\boldsymbol{\nabla}_s$ is the gradient operator in $\boldsymbol{x}_s \equiv (x_s, y_s, z_s)$ coordinates, and

$$\boldsymbol{A} \equiv -\int_{z_s}^{0} \left(\frac{\hat{v}}{f_s}, -\frac{\hat{u}}{f_s}, 0 \right) \mathrm{d}z_s'. \tag{4.13}$$

But

$$-\iiint \mathrm{d}\boldsymbol{x}_s\, \theta \boldsymbol{\nabla}_s \cdot \boldsymbol{A} = \iiint \mathrm{d}\boldsymbol{x}_s\, \boldsymbol{A} \cdot \boldsymbol{\nabla}_s \theta, \tag{4.14}$$

because $\boldsymbol{A}$ vanishes at both boundaries. Then

$$\begin{aligned} \iiint \mathrm{d}\boldsymbol{x}_s\, \boldsymbol{A} \cdot \boldsymbol{\nabla}_s \theta &= \iiint \mathrm{d}\boldsymbol{x}_s\, \boldsymbol{A} \cdot f_s \left(\frac{\partial \hat{v}}{\partial z_s}, -\frac{\partial \hat{u}}{\partial z_s}, 0 \right) \\ &= -\iiint \mathrm{d}\boldsymbol{x}_s \frac{\partial \boldsymbol{A}}{\partial z_s} \cdot f_s(\hat{v}, -\hat{u}, 0) = -\iiint \mathrm{d}\boldsymbol{x}_s (\hat{u}^2 + \hat{v}^2). \end{aligned} \tag{4.15}$$

Collecting all these results, we obtain the Lagrangian for the large-scale semi-geostrophic equations in the form

$$L_{LSG} = \iiint \mathrm{d}\boldsymbol{a} \left\{ -R(x_s, y_s) \frac{\partial x_s}{\partial \tau} + P(x_s, y_s) \frac{\partial y_s}{\partial \tau} + \tfrac{1}{2}\epsilon \bar{\boldsymbol{u}}^2 + \tfrac{1}{2}\epsilon \hat{\boldsymbol{u}}^2 + \theta(\boldsymbol{a})\, z_s + \phi_s \left[\frac{\partial(\boldsymbol{x}_s)}{\partial(\boldsymbol{a})} - \alpha_0 \right] \right\}, \tag{4.16}$$

where $\bar{\boldsymbol{u}} = (\bar{u}, \bar{v})$ and $\hat{\boldsymbol{u}} = (\hat{u}, \hat{v})$. The Lagrangian (4.16) is a functional of the particle locations $\boldsymbol{x}_s(\boldsymbol{a}, \tau)$ and the Lagrange multiplier $\phi_s(\boldsymbol{a}, \tau)$ for the (transformed) incompressibility constraint, through the definitions (4.3) and (4.4). However, unlike the Lagrangian (3.18), (3.19) for the semi-geostrophic equations, (4.16) depends simply on the x_s-coordinates alone. In fact, the original x-coordinates have entirely disappeared from the formulation, except insofar as they will (eventually) be needed to transform the results back into physical space. Even the locations of the rigid boundaries take a simple form in the new coordinates.

The Lagrangian (4.16) for LSGE is formally identical to the Lagrangian for the planetary geostrophic equations (2.6) except for the two $O(\epsilon)$ terms in $\bar{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}$. Yet, as we verify below, the LSGE incorporate inertia with the same accuracy as the semi-geostrophic equations, provided that the internal deformation radius is small. Some of the physics of inertia is taken into account by the transformation from $\boldsymbol{x}$ to $\boldsymbol{x}_s$. This part of the inertia-physics obviously causes nothing more than a geometrical distortion of the flow, and need not even be calculated until it is time to transform the results back to $\boldsymbol{x}$. On the other hand, the ϵ-terms that appear explicitly in (4.16) evidently represent the irreducible part of the inertia-physics – that which cannot be removed by a coordinate transformation – and it is these terms that must be responsible for the qualitatively new behaviour that we expect from the addition of inertia.

To obtain the LSGE, we require that the action based on (4.16) be stationary with respect to variations of $\phi_s(\boldsymbol{a},\tau)$ and the (transformed) fluid-particle trajectories $\boldsymbol{x}_s(\boldsymbol{a},\tau)$. We require that the trajectories and their variations be tangent to the boundaries at $z_s = 0$ and $z_s = -h(x_s, y_s)$. For the trajectory variations, we find that

$$\delta\int L_{LSG}\,\mathrm{d}\tau = \int \mathrm{d}\tau \iiint \mathrm{d}\boldsymbol{a}\left\{-f_s\frac{\partial x_s}{\partial\tau}\delta y_s + f_s\frac{\partial y_s}{\partial\tau}\delta x_s + \theta\delta z_s - \boldsymbol{\nabla}_s\phi\cdot\delta\boldsymbol{x}_s\right\} + \int \mathrm{d}\tau(\delta\bar{K}+\delta\hat{K}), \tag{4.17}$$

where

$$\bar{K} = \iiint \mathrm{d}\boldsymbol{a}\,\tfrac{1}{2}\epsilon\bar{\boldsymbol{u}}^2 \quad\text{and}\quad \hat{K} = \iiint \mathrm{d}\boldsymbol{a}\,\tfrac{1}{2}\epsilon\hat{\boldsymbol{u}}^2. \tag{4.18}$$

In Appendix A we show that

$$\delta\bar{K} = \epsilon\iiint \mathrm{d}\boldsymbol{a}\left\{-\bar{u}\delta\boldsymbol{x}_s\cdot\boldsymbol{\nabla}_s\left(\frac{\partial x_s}{\partial\tau}\right) - \bar{v}\delta\boldsymbol{x}_s\cdot\boldsymbol{\nabla}_s\left(\frac{\partial y_s}{\partial\tau}\right) - \frac{\partial\bar{\boldsymbol{u}}}{\partial\tau}\cdot\delta\boldsymbol{x}_s + \delta\boldsymbol{x}_s\cdot\boldsymbol{\nabla}_s(\tfrac{1}{2}\bar{\boldsymbol{u}}^2)\right\}, \tag{4.19}$$

and in Appendix B we show that

$$\delta\hat{K} = \epsilon\iiint \mathrm{d}\boldsymbol{a}\{\delta\boldsymbol{x}_s\cdot[\boldsymbol{\nabla}_s(\tfrac{1}{2}\hat{\boldsymbol{u}}^2) - (\boldsymbol{\nabla}_s\cdot\boldsymbol{A})\boldsymbol{\nabla}_s\theta]\}. \tag{4.20}$$

Thus

$$\delta\phi_s:\quad \frac{\partial(\boldsymbol{x}_s)}{\partial(\boldsymbol{a})} = \alpha_0, \tag{4.21}$$

and

$$\delta x_s:\quad f\frac{\partial y_s}{\partial\tau} - \frac{\partial\phi_s}{\partial x_s} + \frac{\partial}{\partial x_s}(\tfrac{1}{2}\epsilon\bar{\boldsymbol{u}}^2 + \tfrac{1}{2}\epsilon\hat{\boldsymbol{u}}^2)$$
$$-\epsilon(\boldsymbol{\nabla}_s\cdot\boldsymbol{A})\frac{\partial\theta}{\partial x_s} - \epsilon\bar{u}\frac{\partial}{\partial x_s}\frac{\partial x_s}{\partial\tau} - \epsilon\bar{v}\frac{\partial}{\partial x_s}\frac{\partial y_s}{\partial\tau} - \epsilon\frac{\partial\bar{u}}{\partial\tau} = 0. \tag{4.22}$$

The δy_s- and δz_s-variations yield analogous equations.

Now let

$$(U, V, W) \equiv \frac{\partial}{\partial\tau}(x_s, y_s, z_s) \tag{4.23}$$

be the velocity of massive fluid particles in $\boldsymbol{x}_s$-coordinates, that is, the true velocity in transformed space, and note that $\bar{\boldsymbol{u}}$ is, by its definition (4.3), the vertical average of $\boldsymbol{U} \equiv (U, V)$, but that $\boldsymbol{U} \neq \bar{\boldsymbol{u}}+\hat{\boldsymbol{u}}$. We want to rewrite (4.22) and its analogues to make them easier to compare to the more exact primitive equations. It is convenient to define

$$\phi = \phi_s - \epsilon\bar{\boldsymbol{u}}^2 - \epsilon\hat{\boldsymbol{u}}^2 + \epsilon\bar{\boldsymbol{u}}\cdot(\boldsymbol{U}-\boldsymbol{u}) + \epsilon\theta\boldsymbol{\nabla}_s\cdot\boldsymbol{A}. \tag{4.24}$$

Then, after integrations by parts, (4.22), its analogues, the τ-derivative of (4.21), and $\partial\theta/\partial\tau = 0$ are equivalent to

$$\epsilon\frac{\partial\bar{u}}{\partial t_s} - \left[f_s + \epsilon\frac{\partial\bar{v}}{\partial x_s} - \epsilon\frac{\partial\bar{u}}{\partial y_s}\right]V = -\frac{\partial\Phi}{\partial x_s} + \epsilon\theta\frac{\partial}{\partial x_s}(\boldsymbol{\nabla}_s\cdot\boldsymbol{A}), \tag{4.25a}$$

$$\epsilon\frac{\partial\bar{v}}{\partial t_s} + \left[f_s + \epsilon\frac{\partial\bar{v}}{\partial x_s} - \epsilon\frac{\partial\bar{u}}{\partial y_s}\right]U = -\frac{\partial\Phi}{\partial y_s} + \epsilon\theta\frac{\partial}{\partial y_s}(\boldsymbol{\nabla}_s\cdot\boldsymbol{A}), \tag{4.25b}$$

$$0 = -\frac{\partial\Phi}{\partial z_s} + \theta + \epsilon\theta\frac{\partial}{\partial z_s}(\boldsymbol{\nabla}_s\cdot\boldsymbol{A}), \tag{4.25c}$$

$$\frac{\partial U}{\partial x_s} + \frac{\partial V}{\partial y_s} + \frac{\partial W}{\partial z_s} = 0, \tag{4.25d}$$

$$\frac{\partial\theta}{\partial t_s} + U\frac{\partial\theta}{\partial x_s} + V\frac{\partial\theta}{\partial y_s} + W\frac{\partial\theta}{\partial z_s} = 0, \tag{4.25e}$$

where

$$\Phi \equiv \phi + \tfrac{1}{2}\epsilon(\bar{\boldsymbol{u}}+\hat{\boldsymbol{u}})^2. \tag{4.26}$$

Equations (4.25) are the LSGE in $\boldsymbol{x}_s$-coordinates; $\partial/\partial t_s$ is the time derivative with $\boldsymbol{x}_s$ held fixed. The boundary conditions are no flow through the rigid boundaries at the top and bottom, that is

$$W = 0 \quad \text{at} \quad z_s = 0, \quad \text{and} \quad W = -U\frac{\partial h}{\partial x_s} - V\frac{\partial h}{\partial y_s} \quad \text{at} \quad z_s = -h(x_s, y_s). \tag{4.27}$$

Since Φ is determined by the continuity equation (4.25*d*) and these rigid-boundary conditions, its definition (4.26) is actually irrelevant, but (4.26) makes the planetary geostrophic equations agree with the primitive equations with pressure ϕ, to within an error of $O(\epsilon^2)$.

5. Properties of the large-scale semi-geostrophic equations

First, we verify that (4.25) and (4.26) agree with the primitive equations, within the $O(\epsilon^2)$ error. We have already shown that (4.21), and hence (4.25*d*), is an accurate approximation to the corresponding primitive continuity equation (2.1*c*). We have also shown that the equations for the boundaries in $\boldsymbol{x}_s$-coordinates, and hence (4.27), are formally identical to the corresponding equations in $\boldsymbol{x}$-coordinates, within the $O(\epsilon^2)$ error. Clearly, (4.25*e*) is equivalent to $\partial\theta/\partial\tau = 0$. Therefore, it only remains to check the momentum equations (4.25*a*–*c*). The horizontal momentum equations (4.25*a*, *b*) *seem* to incorporate only the inertia in the vertically averaged flow. However, we shall see that the last terms in (4.25*a*–*c*), and the transformation between $\boldsymbol{x}$ and $\boldsymbol{x}_s$, accurately account for the inertia in $\hat{\boldsymbol{u}}$.

We start with the vertical momentum equation (4.25 c). The first term is

$$-\frac{\partial\Phi}{\partial z_s}=-\frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial z_s}-\frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial z_s}-\frac{\partial\Phi}{\partial z}\frac{\partial z}{\partial z_s}$$

$$=\epsilon\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial z_s}\left(\frac{\hat{v}}{f_s}\right)-\epsilon\frac{\partial\Phi}{\partial y}\frac{\partial}{\partial z_s}\left(\frac{\hat{u}}{f_s}\right)-\frac{\partial\Phi}{\partial z}\left[1+\epsilon\frac{\partial}{\partial x_s}\left(\frac{\hat{v}}{f_s}\right)-\epsilon\frac{\partial}{\partial y_s}\left(\frac{\hat{u}}{f_s}\right)\right]$$

$$=\epsilon\frac{\partial\phi}{\partial x}\frac{\partial}{\partial z}\left(\frac{\hat{v}}{f}\right)-\epsilon\frac{\partial\phi}{\partial y}\frac{\partial}{\partial z}\left(\frac{\hat{u}}{f}\right)$$

$$-\left[\frac{\partial\phi}{\partial z}+\epsilon(\bar{\boldsymbol{u}}+\hat{\boldsymbol{u}})\cdot\frac{\partial}{\partial z}(\bar{\boldsymbol{u}}+\hat{\boldsymbol{u}})\right]\left[1+\epsilon\frac{\partial}{\partial x}\left(\frac{\hat{v}}{f}\right)-\epsilon\frac{\partial}{\partial y}\left(\frac{\hat{u}}{f}\right)\right]+O(\epsilon^2), \quad (5.1)$$

where we have consistently replaced x_s-derivatives by x-derivatives in the $O(\epsilon)$ terms. Thus

$$-\frac{\partial\Phi}{\partial z_s}=-\frac{\partial\phi}{\partial z}\left[1+\epsilon\frac{\partial}{\partial x}\left(\frac{\hat{v}}{f}\right)-\epsilon\frac{\partial}{\partial y}\left(\frac{\hat{u}}{f}\right)\right]+O(\epsilon^2)$$

$$=-\frac{\partial\phi}{\partial z}-\epsilon\theta\left[\frac{\partial}{\partial x}\left(\frac{\hat{v}}{f}\right)-\frac{\partial}{\partial y}\left(\frac{\hat{u}}{f}\right)\right]+O(\epsilon^2)=-\frac{\partial\phi}{\partial z}-\epsilon\theta\frac{\partial}{\partial z}(\nabla\cdot A)+O(\epsilon^2). \quad (5.2)$$

Substituting (5.2) back into (4.25 c), we see that (4.25 c) is equivalent to

$$0=-\frac{\partial\phi}{\partial z}+\theta+O(\epsilon^2). \quad (5.3)$$

The two largest terms in the x_s-momentum equation (4.25 a) are $-f_s V$ and $-\partial\Phi/\partial x_s$. By steps similar to those in (5.1) and (5.2),

$$-\frac{\partial\Phi}{\partial x_s}=-\frac{\partial\phi}{\partial x}-\epsilon(\bar{\boldsymbol{u}}+\hat{\boldsymbol{u}})\cdot\frac{\partial\bar{\boldsymbol{u}}}{\partial x}-\frac{\epsilon}{f}(\bar{\boldsymbol{u}}+\hat{\boldsymbol{u}})\cdot\hat{\boldsymbol{u}}\frac{\partial f}{\partial x}-\epsilon\theta\frac{\partial}{\partial x}(\nabla\cdot A)+O(\epsilon^2). \quad (5.4)$$

Similarly,

$$-f(x_s,y_s)\,V=-\left[f(x,y)+\epsilon\nabla f(x,y)\cdot\left(\frac{\hat{v}}{f},-\frac{\hat{u}}{f}\right)+\dots\right]\frac{\partial}{\partial\tau}\left(y-\epsilon\frac{\hat{u}}{f}\right)$$

$$=-fv-\epsilon\frac{v}{f}\nabla f\cdot(\hat{v},\,-\hat{u})+\epsilon f\frac{\partial}{\partial\tau}\left(\frac{\hat{u}}{f}\right)+O(\epsilon^2). \quad (5.5)$$

Substituting (5.4) and (5.5) into (4.25 a), and noting that all the terms in the derivatives of f cancel to $O(\epsilon^2)$, we eventually obtain

$$\epsilon\frac{\partial}{\partial\tau}(\bar{u}+\hat{u})-fv=-\frac{\partial\phi}{\partial x}+O(\epsilon^2). \quad (5.6)$$

By similar steps, we verify the y_s-direction equation. Thus, the LSGE (4.25)–(4.27) are a consistent $O(\epsilon^2)$ approximation for large-scale nearly geostrophic flow.

The LSGE exactly conserve the energy,

$$E_{LSG}=\iiint d\boldsymbol{x}_s\{\tfrac{1}{2}\epsilon\bar{\boldsymbol{u}}^2-\tfrac{1}{2}\epsilon\hat{\boldsymbol{u}}^2-z_s\,\theta\}, \quad (5.7)$$

and the potential vorticity,

$$Q_{LSG}=\left[f_s+\epsilon\frac{\partial\bar{v}}{\partial x_s}-\epsilon\frac{\partial\bar{u}}{\partial y_s}\right]\frac{\partial\theta}{\partial z_s}, \quad (5.8)$$

on fluid particles, $\partial Q_{LSG}/\partial\tau = 0$. These conservation laws follow from the time- and particle-relabelling symmetries of the Lagrangian (4.16), but can also be proved directly from (4.25). In (5.7), the kinetic energy in the non-vertically-averaged flow *seems* to occur with the wrong sign, and, similarly, (5.8) *seems* to ignore the relative vorticity in the non-vertically-averaged flow. Once again, however, the apparently missing physics is hidden in the transformation between $\boldsymbol{x}$ and $\boldsymbol{x}_s$. In fact, calculations like those given in (5.1) and (5.2) show that

$$E_{LSG} = \iiint \mathrm{d}\boldsymbol{x}\,\{\tfrac{1}{2}\epsilon\bar{\boldsymbol{u}}^2 + \tfrac{1}{2}\epsilon\hat{\boldsymbol{u}}^2 - z\theta\} + O(\epsilon^2), \tag{5.9}$$

and

$$Q_{LSG} = f\frac{\partial\theta}{\partial z} + \epsilon\frac{\partial\theta}{\partial z}\left[\frac{\partial}{\partial x}(\bar{v}+\hat{v}) - \frac{\partial}{\partial y}(\bar{u}+\hat{u})\right] - \epsilon\frac{\partial\theta}{\partial x}\frac{\partial(\bar{v}+\hat{v})}{\partial z} + \epsilon\frac{\partial\theta}{\partial y}\frac{\partial(\bar{u}+\hat{u})}{\partial z} + O(\epsilon^2). \tag{5.10}$$

Thus the conserved energy and potential vorticity are consistent-order approximations to the energy (2.4) and potential vorticity (2.5) of the primitive equations.

We get a further impression of the physical content of the LSGE by examining solutions of the linearized equations. Suppose that the depth is uniform, $h = h_0$ (constant). Then the equations governing small departures from a state of rest with flat isothermal surfaces are

$$\epsilon\frac{\partial\bar{u}}{\partial t_s} - f_s V = -\frac{\partial\Phi'}{\partial x_s}, \quad \epsilon\frac{\partial\bar{v}}{\partial t_s} + f_s U = -\frac{\partial\Phi'}{\partial y_s}, \tag{5.11a, b}$$

$$0 = -\frac{\partial\Phi'}{\partial z_s} + \theta - \epsilon N^2(\boldsymbol{\nabla}_s\cdot\boldsymbol{A}), \tag{5.11c}$$

$$\frac{\partial U}{\partial x_s} + \frac{\partial V}{\partial y_s} + \frac{\partial W}{\partial z_s} = 0, \tag{5.11d}$$

$$\frac{\partial\theta}{\partial t_s} + WN^2 = 0, \tag{5.11e}$$

where

$$\Phi' \equiv \Phi - \epsilon\theta\boldsymbol{\nabla}_s\cdot\boldsymbol{A}, \tag{5.12}$$

and N is the Brunt–Väisälä frequency, assumed constant for convenience. The boundary conditions are $W = 0$ at $z_s = 0, -h_0$. Remember that $(\bar{u}, \bar{v})$ are the vertical averages of the true fluid-particle velocities (U, V).

Solutions of (5.11) are separable in the form

$$(U, V, \Phi') = \sum_{m=0}^{\infty} (U_m, V_m, \Phi'_m)(x_s, y_s)\cos\left(\frac{m\pi z_s}{h_0}\right) \tag{5.13}$$

and

$$(W, \theta) = \sum_{m=1}^{\infty} (W_m, \theta_m)(x_s, y_s)\sin\left(\frac{m\pi z_s}{h_0}\right). \tag{5.14}$$

We find that

$$\boldsymbol{\nabla}_s\cdot\boldsymbol{A} = \sum_{m=1}^{\infty}\frac{h_0^2}{m^2\pi^2}\boldsymbol{\nabla}_s\cdot\left(\frac{1}{f^2}\boldsymbol{\nabla}_s\theta_m\right)\sin\left(\frac{m\pi z_s}{h_0}\right). \tag{5.15}$$

The $m = 0$ (i.e. external mode) equations take the form

$$\epsilon\frac{\partial U_0}{\partial t_s} - f_s V_0 = -\frac{\partial\Phi'_0}{\partial x_s}, \quad \epsilon\frac{\partial V_0}{\partial t_s} + f_s U_0 = -\frac{\partial\Phi'_0}{\partial y_s}, \quad \frac{\partial U_0}{\partial x_s} + \frac{\partial V_0}{\partial y_s} = 0, \tag{5.16a–c}$$

and the $m \neq 0$ (i.e. internal mode) equations take the form

$$-f_s V_m = -\frac{\partial \Phi'_m}{\partial x_s}, \quad f_s U_m = -\frac{\partial \Phi'_m}{\partial y_s}, \tag{5.17a, b}$$

$$0 = \frac{m\pi}{h_0}\Phi'_m + \theta_m - \epsilon \frac{N^2 h_0^2}{m^2\pi^2}\nabla_s \cdot \left(\frac{1}{f_s^2}\nabla_s \theta_m\right), \tag{5.17c}$$

$$\frac{\partial U_m}{\partial x_s} + \frac{\partial V_m}{\partial y_s} + \frac{m\pi}{h_0} W_m = 0, \tag{5.17d}$$

$$\frac{\partial \theta}{\partial t_s} + N^2 W_m = 0. \tag{5.17e}$$

Inertia is present in both (5.16) and (5.17), but in (5.17) it is hidden in the last term of (5.17*c*).

Now suppose that $f_s = f_0 + \beta y_s$, in the usual beta-plane approximation. Then the linearized primitive equations have (WKB) plane-wave solutions satisfying the Rossby-wave dispersion relation

$$\omega = \frac{-\beta k \lambda_m^2}{1 + \epsilon(k^2 + l^2)\lambda_m^2}, \tag{5.18}$$

where ω is the frequency, (k, l) is the horizontal wavevector, and $\lambda_m \equiv N h_0 / m\pi f$ is the deformation radius corresponding to mode m. Note that $\lambda_0 = \infty$ on account of the rigid lid. The plane-wave solutions of (5.16) have dispersion relation

$$\omega = \frac{-\beta k}{\epsilon(k^2 + l^2)}, \tag{5.19}$$

in exact agreement with (5.18) when $m = 0$. However, the plane-wave solutions of (5.17) have dispersion relation

$$\omega = -\beta k \lambda_m^2 [1 - \epsilon \lambda_m^2 (k^2 + l^2)], \tag{5.20}$$

which represents the first two terms in an expansion of (5.18), assuming that $\lambda_m^2(k^2 + l^2) \ll 1$, that is, assuming horizontal lengthscales larger than the internal deformation radii.

To solve the LSGE in their general nonlinear form (4.25), it is convenient to form a vorticity equation for the vertically integrated flow. By the continuity equation (4.25*d*) and boundary conditions (4.27),

$$\frac{\partial}{\partial x_s}(h_s \bar{u}) + \frac{\partial}{\partial y_s}(h_s \bar{v}) = 0. \tag{5.21}$$

Hence

$$h_s \bar{u} = -\frac{\partial \psi}{\partial y_s} \quad \text{and} \quad h_s \bar{v} = \frac{\partial \psi}{\partial x_s} \tag{5.22}$$

for some $\psi(x_s, y_s, t_s)$. The horizontal momentum equations (4.25*a, b*) become

$$-\epsilon \frac{1}{h_s}\frac{\partial^2 \psi}{\partial y_s \partial t_s} - \left[f_s + \epsilon \nabla_s \cdot \left(\frac{1}{h_s}\nabla_s \psi\right)\right] V = -\frac{\partial \Phi}{\partial x_s} + \epsilon\theta \frac{\partial}{\partial x_s}\nabla_s \cdot \boldsymbol{A}, \tag{5.23a}$$

$$+\epsilon \frac{1}{h_s}\frac{\partial^2 \psi}{\partial x_s \partial t_s} + \left[f_s + \epsilon \nabla_s \cdot \left(\frac{1}{h_s}\nabla_s \psi\right)\right] U = -\frac{\partial \Phi}{\partial y_s} + \epsilon\theta \frac{\partial}{\partial y_s}\nabla_s \cdot \boldsymbol{A}. \tag{5.23b}$$

We take the vertical integrals of (5.23), using (4.25*c*) to write

$$\int_{-h_s}^{0} dz_s \frac{\partial \Phi}{\partial x_s} = \Phi(x_s, y_s, -h_s)\frac{\partial h_s}{\partial x_s} - \epsilon \frac{\partial}{\partial x_s}\int_{-h_s}^{0} dz_s\, z_s\left[\theta + \theta\frac{\partial}{\partial z_s}\nabla_s \cdot A\right], \text{etc.,} \tag{5.24}$$

divide the results by h_s, and cross-differentiate to remove Φ. The result is a vorticity equation for the vertically averaged flow,

$$\epsilon\nabla_s\cdot\left(\frac{1}{h_s}\nabla_s\frac{\partial\psi}{\partial t_s}\right) + J\left(\psi, \frac{f+\epsilon\nabla_s\cdot\left(\frac{1}{h_s}\nabla_s\psi\right)}{h_s}\right) = J\left(\frac{1}{h_s}, \int_{-h_s}^{0} dz_s\, z_s\left[\theta - \epsilon\frac{\partial\theta}{\partial z_s}\nabla_s\cdot A\right]\right)$$
$$-\epsilon\frac{\partial}{\partial x_s}\left[\int_{-h_s}^{0}\frac{1}{h_s}\frac{\partial\theta}{\partial y_s}(\nabla_s\cdot A)\,dz_s\right] + \epsilon\frac{\partial}{\partial y_s}\left[\int_{-h_s}^{0}\frac{1}{h_s}\frac{\partial\theta}{\partial x_s}(\nabla_s\cdot A)\,dz_s\right]. \tag{5.25}$$

Here,

$$J(F, G) \equiv \frac{\partial(F, G)}{\partial(x_s, y_s)} \tag{5.26}$$

is the horizontal Jacobian operator in $\mathbf{x}_s$-coordinates. Note that the right-hand side of (5.25) depends only on the temperature $\theta(\mathbf{x}_s, t_s)$, and not on $\psi(x_s, y_s, t_s)$.

Now suppose we know $\psi(x_s, y_s, t_s)$ and $\theta(\mathbf{x}_s, t_s)$ at some initial time. Then we can determine the velocity field (U, V, W) at that time as follows. The horizontal components are

$$U = -\frac{1}{h_s}\frac{\partial\psi}{\partial y_s} + \hat{U} \quad\text{and}\quad V = \frac{1}{h_s}\frac{\partial\psi}{\partial x_s} + \hat{V}, \tag{5.27}$$

where $(\hat{U}, \hat{V})$, the departure of the true velocity field from its vertical average, is determined from ψ and θ by the vertical derivatives of (5.23*a*, *b*), and by the requirement

$$\int_{-h_s}^{0} dz_s(\hat{U}, \hat{V}) = 0. \tag{5.28}$$

Then

$$W = \int_{z_s}^{0}\left[\frac{\partial U}{\partial x_s} + \frac{\partial V}{\partial y_s}\right] dz_s'. \tag{5.29}$$

With the velocity field thus determined, we use (4.25*e*) to step θ forward to a new time. To determine ψ at the new time, we first solve (5.25), a linear elliptic equation for $\partial\psi/\partial t_s$, subject to the boundary condition $\partial\psi/\partial t_s = 0$ at the coastal boundary, where $h_s = 0$. Since $(\hat{u}, \hat{v})$ vanish with h_s, this boundary has the same location in $\mathbf{x}_s$-space as in $\mathbf{x}$-space. With $\partial\psi/\partial t_s$ thus determined, we step ψ, and the cycle is complete. This algorithm fails if (and only if) the coefficients of U and V in (5.23*a*, *b*) vanish. Hence, the condition for LSGE to be well-posed is evidently

$$\tilde{f} \equiv f_s + \epsilon\nabla_s\cdot\left(\frac{1}{h_s}\nabla_s\psi\right) > 0, \tag{5.30}$$

the condition for symmetric stability of the vertically averaged flow. The condition (5.30) is far milder than the two corresponding conditions in the semi-geostrophic equations (which must be enforced at each three-dimensional location).

Interestingly, the LSGE filter out the inertia–gravity waves in three space dimensions, but only require the solution of a two-dimensional elliptic equation (5.25). (In contrast, the quasi-geostrophic and semi-geostrophic equations both require the solution of a three-dimensional elliptic equation.) However, (5.20) shows that the group and phase velocities of the internal Rossby waves become infinite as the wavelengths vanish (violating our requirement that horizontal lengthscales be larger than the internal deformation radii). In the limit of perfect spatial resolution, these waves carry information at infinite speed, mimicking the behaviour of a three-dimensional elliptic equation. These facts suggest that the LSGE will be most useful as the basis for numerical models, in which the spatial resolution is always limited by the grid size. Even the largest existing numerical ocean circulation models barely resolve the internal deformation radius.

With ϵ set formally to zero (and thus ignoring the distinction between $\boldsymbol{x}$ and $\boldsymbol{x}_s$) the LSGE reduce to the planetary geostrophic equations (2.6), and it is clear that the solution algorithm given above is a straightforward generalization of the algorithm discussed by Salmon (1994) for the planetary geostrophic equations. However, when ϵ vanishes, (5.25) changes type, from an elliptic equation for $\partial\psi/\partial t$, to a hyperbolic equation for ψ. The hyperbolic equation is obviously incapable of satisfying the boundary conditon $\psi = 0$, and, as emphasized by Salmon (1994), the planetary geostrophic equations therefore require horizontal friction terms to make the analogue of (5.25) elliptic.

Salmon (1994) used the ansatz (2.13) to reduce the planetary geostrophic equations to a pair of coupled two-dimensional equations, which then formed the basis for a very simple and efficient numerical ocean-circulation model. Because the LSGE potential vorticity (5.8) takes such a simple form in $\boldsymbol{x}_s$-coordinates, a similar reduction applies to the LSGE. By the same reasoning as in §2, the ansatz

$$\theta = F''\left(\frac{z_s}{\tilde{f}} + S(x_s, y_s, t_s)\right) \tag{5.31}$$

must be consistent with the LSGE. Here, F is an arbitrary function, and the primes, which denote differentiation, are introduced for later convenience; $\tilde{f}$ is defined by (5.30); and S is a z_s-independent function to be determined by substitution back into the LSGE. One eventually obtains a coupled pair of equations for $\partial\psi/\partial t_s$ and $\partial S/\partial t_s$. The ψ-equation is just (5.25) with (5.31) substituted into its right-hand side. To obtain the S-equation, we substitute (5.31) and the resulting expressions (in terms of ψ and S) for the velocity (U, V, W) into the temperature equation (4.25e), verifying that the many z_s-dependent terms cancel out. However, it is much easier to obtain this S-equation by simply evaluating the temperature equation at $z_s = 0$ (where $W = 0$). We obtain

$$h_s \frac{\partial S}{\partial t_s} + J(\psi, S) + \int_{-h_s}^{0} (h_s + z_s)\left[\frac{\partial S}{\partial x_s}\frac{\partial U}{\partial z_s} + \frac{\partial S}{\partial y_s}\frac{\partial V}{\partial z_s}\right] \mathrm{d}z_s = 0, \tag{5.32}$$

where J is defined by (5.26). The second term in (5.32) represents the temperature advection by the vertically averaged flow, and the last term represents the advection by the remaining part of the flow. Then, further eliminating (U, V) in favour of ψ and S, and omitting some tedious details, we eventually obtain

$$h_s \frac{\partial S}{\partial t_s} + J(\psi, S) + cJ(S, \tilde{f}) - \epsilon\alpha J\left(S, \frac{\tilde{f}}{h_s}\right) = 0, \tag{5.33}$$

where

$$c \equiv \frac{1}{\tilde{f}^2} \int_{-h_s}^{0} (h_s + z_s)\, z_s \frac{\partial \theta}{\partial z_s} \mathrm{d}z_s \tag{5.34}$$

is the phase speed of internal Rossby waves, and

$$\alpha \equiv \frac{h_s}{\tilde{f}} \int_{-h_s}^{0} (\nabla_s \cdot A) z_s \frac{\partial \theta}{\partial z_s} \mathrm{d}z_s. \tag{5.35}$$

The coefficients (5.34) and (5.35) can, in turn, be written as lengthy but explicit functions of ψ, S, and their derivatives, involving the arbitrary function F. For example,

$$ch_s = 2\left[F(S) - F\left(S - \frac{h_s}{\tilde{f}}\right)\right] - \frac{h_s}{\tilde{f}}\left[F'(S) + F'\left(S - \frac{h_s}{\tilde{f}}\right)\right]. \tag{5.36}$$

If we choose the arbitrary function $F''(\,)$ to be a step function, then these equations reduce to the equations obtained by making the LSGE approximations on a system composed of two homogeneous layers. But whatever the choice for $F(\,)$, the solutions of the two-dimensional equations are also solutions of the full three-dimensional LSGE.

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Appendix A. Derivation of (4.19)

It is easiest to consider variations $\delta \boldsymbol{a}(\boldsymbol{x}_s, t)$ in the inverse mapping from $\boldsymbol{x}_s$-space back to $\boldsymbol{a}$-space. Then

$$\begin{aligned}\delta \bar{K} &= \epsilon \delta \iiint \mathrm{d}\boldsymbol{x}_s \frac{\partial(\boldsymbol{a})}{\partial(\boldsymbol{x}_s)} \tfrac{1}{2}\bar{\boldsymbol{u}}^2 \\ &= \epsilon \iiint \mathrm{d}\boldsymbol{x}_s \left[\tfrac{1}{2}\bar{\boldsymbol{u}}^2 \delta \frac{\partial(\boldsymbol{a})}{\partial(\boldsymbol{x}_s)} + \frac{\partial(\boldsymbol{a})}{\partial(\boldsymbol{x}_s)} \bar{\boldsymbol{u}} \cdot \delta \bar{\boldsymbol{u}}\right] \\ &= \epsilon \iiint \mathrm{d}\boldsymbol{a} [\delta \boldsymbol{x}_s \cdot \nabla_s (\tfrac{1}{2}\bar{\boldsymbol{u}}^2) + \bar{\boldsymbol{u}} \cdot \delta \bar{\boldsymbol{u}}]. \end{aligned} \tag{A 1}$$

We go to work on the last term in (A 1). Adopting the careful notation of §3,

$$\bar{u}(x_0, y_0, t) \equiv \frac{1}{h(x_0, y_0)} \int_{-h(x_0, y_0)}^{0} \frac{\partial x_s}{\partial \tau} (\boldsymbol{a}(x_0, y_0, z, t; \mu), t; \mu) \, \mathrm{d}z. \tag{A 2}$$

Thus

$$\begin{aligned}\delta \bar{u} \equiv \frac{\partial \bar{u}}{\partial \mu} &= \frac{1}{h} \int_{-h}^{0} \left[\frac{\partial}{\partial \boldsymbol{a}}\left(\frac{\partial x_s}{\partial \tau}\right) \cdot \delta \boldsymbol{a} + \frac{\partial}{\partial \tau}(\delta x_s)\right] \mathrm{d}z \\ &= \frac{1}{h} \int_{-h}^{0} \left[-\nabla_s \frac{\partial x_s}{\partial \tau} \cdot \delta \boldsymbol{x}_s + \frac{\partial}{\partial \tau}(\delta x_s)\right] \mathrm{d}z. \end{aligned} \tag{A 3}$$

Hence

$$\begin{aligned}\iiint \mathrm{d}\boldsymbol{a}\, \bar{u} \delta \bar{u} &= \iint \mathrm{d}x_s \, \mathrm{d}y_s \, h \bar{u} \delta \bar{u} \\ &= \iiint \mathrm{d}\boldsymbol{x}_s \bar{u} \left[-\nabla_s \frac{\partial x_s}{\partial \tau} \cdot \delta \boldsymbol{x}_s + \frac{\partial \delta x_s}{\partial \tau}\right] \\ &= \iiint \mathrm{d}\boldsymbol{x}_s \left[-\bar{u} \nabla_s \frac{\partial x_s}{\partial \tau} \cdot \delta \boldsymbol{x}_s - \frac{\partial \bar{u}}{\partial \tau} \delta x_s\right]. \end{aligned} \tag{A 4}$$

Similarly,

$$\iiint \mathrm{d}\boldsymbol{a}\,\bar{v}\delta\bar{v} = \iiint \mathrm{d}\boldsymbol{x}_s\left[-\bar{v}\boldsymbol{\nabla}_s\frac{\partial y_s}{\partial \tau}\cdot\delta\boldsymbol{x}_s - \frac{\partial \bar{v}}{\partial \tau}\delta y_s\right]. \tag{A 5}$$

Then, combining (A 1), (A 4) and (A 5), we obtain (4.19).

Appendix B. Derivation of (4.20)

Again, it is much easier to consider variations $\delta\boldsymbol{a}(\boldsymbol{x}_s, t)$ in the inverse mapping. By steps similar to those in Appendix A,

$$\delta\hat{K} = \epsilon\iiint \mathrm{d}\boldsymbol{a}[\delta\boldsymbol{x}_s \aleph \boldsymbol{\nabla}_s(\tfrac{1}{2}\hat{\boldsymbol{u}}^2) + \hat{\boldsymbol{u}}\cdot\delta\hat{\boldsymbol{u}}]. \tag{B 1}$$

By its definition (4.4),

$$\delta\hat{u} = \delta\left[-\frac{1}{f_s}\int_{-h_s}^{z_s}\frac{\partial\theta}{\partial y_s}\mathrm{d}z_s - \frac{1}{f_s}\int_{-h_s}^{0}\frac{z_s}{h_s}\frac{\partial\theta}{\partial y_s}\mathrm{d}z_s\right]$$

$$= -\frac{1}{f_s}\int_{-h_s}^{z_s}\frac{\partial\delta\theta}{\partial y_s}\mathrm{d}z_s - \frac{1}{f_s}\int_{-h_s}^{0}\frac{z_s}{h_s}\frac{\partial\delta\theta}{\partial y_s}\mathrm{d}z_s. \tag{B 2}$$

However, because the vertical integral of $\hat{u}$ vanishes, the last (z_s-independent) term in (B 2) makes no contribution to (B 1). Thus

$$\iiint \mathrm{d}\boldsymbol{x}_s\,\hat{u}\delta\hat{u} = -\iint \mathrm{d}x_s\,\mathrm{d}y_s\int_{-h_s}^{0}\mathrm{d}z_s\,\hat{u}\int_{-h_s}^{z_s}\frac{1}{f_s}\frac{\partial\delta\theta}{\partial y_s}\mathrm{d}z'_s. \tag{B 3}$$

Interchanging the order of the last two integrations, we have

$$\iiint \mathrm{d}\boldsymbol{x}_s\,\hat{u}\delta\hat{u} = -\iint \mathrm{d}x_s\,\mathrm{d}y_s\int_{-h_s}^{0}\mathrm{d}z_s\frac{1}{f_s}\frac{\partial\delta\theta}{\partial y_s}\int_{z_s}^{0}\hat{u}\,\mathrm{d}z'_s$$

$$= \iiint \mathrm{d}\boldsymbol{x}_s\left[\delta\theta\int_{z_s}^{0}\frac{\partial}{\partial y_s}\left(\frac{\hat{u}}{f_s}\right)\mathrm{d}z'_s\right] = -\iiint \mathrm{d}\boldsymbol{x}_s\left[\boldsymbol{\nabla}_s\theta\cdot\delta\boldsymbol{x}_s\int_{z_s}^{0}\frac{\partial}{\partial y_s}\left(\frac{\hat{u}}{f_s}\right)\mathrm{d}z'_s\right]. \tag{B 4}$$

Similarly,

$$\iiint \mathrm{d}\boldsymbol{x}_s\,\hat{v}\delta\hat{v} = +\iiint \mathrm{d}\boldsymbol{x}_s\left[\boldsymbol{\nabla}_s\theta\cdot\delta\boldsymbol{x}_s\int_{z_s}^{0}\frac{\partial}{\partial x_s}\left(\frac{\hat{v}}{f_s}\right)\mathrm{d}z'_s\right]. \tag{B 5}$$

Combining (B 1), (B 4) and (B 5), we obtain (4.20).

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