Analogous formulation of electrodynamics and two-dimensional fluid dynamics

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A single, simply stated approximation transforms the equations for a two-dimensional perfect fluid into a form that is closely analogous to Maxwell's equations in classical electrodynamics. All the fluid conservation laws are retained in some form. Waves in the fluid interact only with vorticity and not with themselves. The vorticity is analogous to electric charge density, and point vortices are the analogs of point charges. The dynamics are equivalent to an action principle in which a set of fields and the locations of the point vortices are varied independently. We recover classical, incompressible, point vortex dynamics as a limiting case. Our full formulation represents the generalization of point vortex dynamics to the case of compressible flow.

1. Introduction

This paper offers an approximation to the equations for a two-dimensional perfect fluid that cleanly separates waves from vorticity. In this approximation, waves in the fluid interact only with vorticity and not with themselves. Thus the fluid waves are analogous to electromagnetic waves, and vorticity plays the role of electric charge. If the dependent variables include a carefully chosen set of potentials, then the fluid equations are closely analogous to the equations of classical electrodynamics. The gauge symmetry corresponding to charge conservation is analogous to a gauge symmetry corresponding to vorticity conservation.

Our beginning point will be the non-rotating shallow water equations but, as will be seen, it could be any two-dimensional polytropic fluid. The shallow water equations are

$$\mathbf{u}_t + \zeta(-v, u) = -\nabla(gh + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}), \tag{1.1}$$

$$h_t = -\nabla\cdot(h\mathbf{u}), \tag{1.2}$$

where $h(\mathbf{x}, t)$ is the fluid depth at location $\mathbf{x} = (x, y)$ and time t, $\mathbf{u} = (u, v)$ is the velocity, g is the gravity constant, and $\zeta = v_x - u_y$ is the vorticity. Subscripts denote partial derivatives and $\nabla = (\partial_x, \partial_y)$. For the purpose of making our approximation, it is convenient to express shallow-water dynamics in the equivalent Hamiltonian form,

$$\frac{dF}{dt} = \{F, H\}, \tag{1.3}$$

where $F[u, v, h]$ is an arbitrary functional of $\mathbf{u}(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$,

$$\{A, B\} = \iint d\mathbf{x} \left(q \left(\frac{\delta A}{\delta u}\frac{\delta B}{\delta v} - \frac{\delta B}{\delta u}\frac{\delta A}{\delta v} \right) - \frac{\delta A}{\delta \mathbf{u}}\cdot\nabla\frac{\delta B}{\delta h} + \frac{\delta B}{\delta \mathbf{u}}\cdot\nabla\frac{\delta A}{\delta h} \right), \tag{1.4}$$

is the Poisson bracket (defined for any two functionals A and B), $q = \zeta/h$ is the potential vorticity, and

$$H = \iint d\mathbf{x}\, \tfrac{1}{2}\left(hu^2 + hv^2 + gh^2\right) \tag{1.5}$$

is the Hamiltonian.

Our approximation is a simple one: We replace the exact Hamiltonian (1.5) by the approximate Hamiltonian

$$H_{approx} = \iint d\mathbf{x}\, \tfrac{1}{2}\left(h_0u^2 + h_0v^2 + gh^2\right), \tag{1.6}$$

where h_0 is the constant mean depth. This approximation appears justified in the limit $c \equiv \sqrt{gh_0} \to \infty$ of low Froude number. As we shall see, it cleanly separates waves from vorticity. The bracket (1.4) conserves the Casimirs $\iint d\mathbf{x}\, h\, G(q)$ for any choice of Hamiltonian. Here $G(q)$ is an arbitrary function of the potential vorticity. The Casimirs include mass, mean vorticity, and potential enstrophy. Since our approximation involves only the Hamiltonian, it preserves all of these Casimir invariants as well as the momentum $\iint d\mathbf{x}\, h\mathbf{u}$. It conserves the energy (1.6) instead of (1.5). The retention of conservation laws is a primary advantage of approximation methods based on a Hamiltonian formulation. For reviews of Hamiltonian fluid dynamics, see Salmon (1988) and Shepherd (1990).

The dynamics resulting from (1.4) and (1.6) are

$$\mathbf{u}_t + c^2\nabla\alpha = q(v, -u), \tag{1.7}$$

$$\alpha_t + \nabla\cdot\mathbf{u} = 0, \tag{1.8}$$

where now $\alpha \equiv h/h_0$ and $q = \zeta/\alpha$. It follows from (1.7) and (1.8) that

$$q_t + (\mathbf{u}/\alpha)\cdot\nabla q = 0. \tag{1.9}$$

If we regard $\mathbf{u}/\alpha$ as the 'true' fluid velocity then (1.9) and the continuity equation (1.8) become 'exact', but the 'error' appears elsewhere in the equations. This illustrates a phenomenon that might be called 'identification error.' Dynamical approximations correspond to replacing one set of equations by another, slightly different set. However there always remains the question of how to identify the 'measurements' in one system with those in the other, and this identification is unavoidably ambiguous. Our approximation takes its simplest form when the dependent variables are chosen to be the ones that appear in (1.7) and (1.8), but it is a matter of interpretation whether $\mathbf{u}$ represents velocity or something slightly different. The constant wave speed c is likewise open to interpretation. If the 'parent dynamics' are shallow water dynamics, then $c = \sqrt{gh_0}$ as stated above. However, if we had instead begun from the equations for a two-dimensional compressible fluid, then c would be the sound speed. In the point of view of this paper, the interpretation of c hardly matters; we study (1.7)-(1.8) as a prototypical model for the interaction between waves and vorticity. It only matters that c be large enough (Froude or Mach number small enough) to justify our fundamental approximation of replacing the exact Hamiltonian of the parent model by an expression quadratic in the fluid variables.

We shall refer to the dynamics (1.7)-(1.8) as *wave-vortex* (WV) dynamics. As a prelude to further developments, we note that WV dynamics can usefully be split into two parts which can be considered to operate alternately, each for a single 'time step'. The 'wave split'

$$\begin{aligned} \mathbf{u}_t &= -c^2\nabla\alpha, \\ \alpha_t &= -\nabla\cdot\mathbf{u}, \end{aligned} \tag{1.10}$$

corresponds to linear waves propagating at the constant speed c. The 'vorticity split',

$$\mathbf{u}_t = q(v, -u),$$
$$\alpha_t = 0, \tag{1.11}$$

corresponds to a rotation of the velocity vector at angular speed q. If $q = 0$ the second split is absent and the dynamics consist solely of non-dispersive, non-interacting waves.

Salmon (2009; hereafter S09) showed how the splitting (1.10)-(1.11) could be made the basis of an efficient numerical model. (S09 considered the three-dimensional version of (1.10)-(1.11) and included an additional split corresponding to buoyancy forcing.) A key idea was to further split (1.10) into its direction components and to solve each of these exactly by the method of Riemann invariants. Exact solution of the Riemann splits requires that $c = \Delta x/\Delta t$ where Δx is the grid spacing and Δt is the time step. This scheme works well for the case of vorticity confined to a compact region of the infinite plane; the radiation condition is satisfied by allowing the outward propagating Riemann invariants to escape to infinity. To accommodate coordinate-system rotation, we merely add the Coriolis parameter f to the numerator of q.

In the modeling work described by S09, the solution of WV dynamics was viewed as a method of approximately solving the equations for an exactly incompressible fluid, and the waves (whose speed c could be made arbitrarily large by reducing the time step Δt) were viewed merely as a device for avoiding the solution of elliptic equations in irregular domains. S09 even includes the statement that " ... one would never use our model to study sound waves any more than one would use the lattice Boltzmann method for that purpose." In the present paper we ignore that admonishment, and we study WV dynamics as a prototypical model for the interaction between waves and vorticity.

Our primary motivation is the powerful analogy between (1.7)-(1.8) and the equations of classical electrodynamics. In this analogy, the split (1.10) corresponds to electromagnetic waves, and the potential vorticity q corresponds to electric charge. If charge is absent, then the waves are linear waves that never interact. Phenomena corresponding to wave steepening and shock formation do not occur. In contrast to the exact shallow water equations, nonlinear behavior requires the presence of vorticity.

Before pursuing the analogy with electrodynamics, we pause to consider the one-dimensional form of our dynamics. Suppose that everything is independent of y. Then $\alpha_t = -u_x$ is satisfied if

$$(\alpha, u) = (a_x,\ -a_t) \tag{1.12}$$

for some $a(x,t)$. It then follows from $v_t = -qu$ that $v(x,t) = V(a)$ for some function V. By $q = v_x/a_x$ we conclude that $q = V'(a)$. Then finally $u_t = qv - c^2\alpha_x$ becomes

$$a_{tt} - c^2 a_{xx} = -V'(a)V(a) = -\frac{1}{2}\frac{d}{da}V(a)^2, \tag{1.13}$$

which is a quasi-linear equation for $a(x,t)$. The function $V(a)$ is set by the initial conditions. The form of (1.13), in which the nonlinearity enters only through the 'source term' on the rhs, which contains no derivatives of $a(x,t)$, contrasts sharply with the corresponding form of the shallow water equations, which contains terms nonlinear in the derivatives. These terms cause singularities (shocks) in finite time: the shallow-water equations are ill-posed in the absence of viscosity. However, no such restriction applies to (1.13). It follows from (1.12) that $a_t + (u/\alpha)a_x = 0$. Thus (cf. (1.9)) $a(x,t)$ is advected in the same way as $q(x,t)$ and hence may be considered a particle label. This provides the easiest way of seeing that $q = V'(a)$ for some $V(a)$. In rotating coordinates, and in the linear limit, the rhs of (1.13) equals $-f^2 a$, where f is the Coriolis parameter, and

(1.13) reduces to the Klein-Gordan equation. In the remainder of this paper we ignore coordinate system rotation.

The one-dimensional equation (1.13) is equivalent to the requirement that

$$\iint dx\, dt\, \tfrac{1}{2}\left[a_t^2 - c^2 a_x^2 - V(a)^2\right] \tag{1.14}$$

be stationary with respect to arbitrary variations $\delta a(x,t)$, where, again, $V(a)$ is determined by initial conditions. The remainder of this paper can be viewed as the generalization of (1.13) and (1.14) to two space dimensions. This generalization requires that the vorticity be concentrated at points. If (1.8) is satisfied in a manner similar to (1.12), then the whole formulation closely resembles that of classical electrodynamics.

2. Analogy with electrodynamics

We return to the two-dimensional form of our equations, (1.7)-(1.8). The key step is to realize that (1.8) is analogous to the homogeneous subset of Maxwell's equations, and may be automatically satisfied by the proper choice of potentials. That is, (1.8) is equivalent to the statement

$$(\partial_t, \partial_x, \partial_y) \cdot (\alpha, u, v) = 0 \tag{2.1}$$

that the three-dimensional spacetime divergence of the vector (α, u, v) must vanish. This condition is automatically satisfied by the potential representation

$$(\alpha, u, v) = (\partial_t, \partial_x, \partial_y) \times (\phi, A, B) = (B_x - A_y,\ \phi_y - B_t,\ A_t - \phi_x), \tag{2.2}$$

where $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t) = (A, B)$ and are analogs of the scalar and vector potentials in electrodynamics. The representation (2.2) is analogous to (1.12). Since the curl of a gradient vanishes, the potential representation (2.2) is not unique; the substitutions

$$\phi \to \phi + \lambda_t, \quad \mathbf{A} \to \mathbf{A} + \nabla\lambda, \tag{2.3}$$

cause no change in the physical variables (α, u, v). With (1.8) automatically satisfied, (1.7) takes the form

$$(\partial_{tt} - c^2\nabla^2)\mathbf{A} - \nabla(\phi_t - c^2\nabla\cdot\mathbf{A}) = -q\mathbf{u}. \tag{2.4}$$

We use the gauge arbitrariness (2.3) to require that

$$\phi_t - c^2\nabla\cdot\mathbf{A} = 0. \tag{2.5}$$

Then (2.4) takes the form

$$(\partial_{tt} - c^2\nabla^2)\mathbf{A} = -q\mathbf{u}. \tag{2.6}$$

By the definition of vorticity

$$\zeta = v_x - u_y = (A_t - \phi_x)_x - (\phi_y - B_t)_y = c^{-2}(\partial_{tt} - c^2\nabla^2)\phi, \tag{2.7}$$

after use of the gauge condition (2.5).

To make further progress in the direction of electrodynamics we must introduce the analog of point charges. We assume that the vorticity takes the form

$$\zeta = \sum_i \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t)). \tag{2.8}$$

This assumption is, as it must be, consistent with the conservation of $\iint d\mathbf{x}\ \zeta$, which is one of our conserved Casimirs. Since the point vortices represent singularities in the potential vorticity q, they must, according to (1.9), move at the velocity $\mathbf{u}/\alpha$. Thus

$$\dot{\mathbf{x}}_i(t) = \frac{\mathbf{u}(\mathbf{x}_i(t), t)}{\alpha(\mathbf{x}_i(t), t)}. \tag{2.9}$$

From (2.8) and (2.9) it follows that

$$q\mathbf{u} = \zeta \frac{\mathbf{u}}{\alpha} = \sum_i \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t)) \dot{\mathbf{x}}_i(t) \tag{2.10}$$

depends only on the trajectories of the point vortices. Thus (2.6) and (2.7) may be written in the forms

$$(\partial_{tt} - c^2 \nabla^2)\phi = c^2 \sum_i \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t)), \tag{2.11}$$

$$(\partial_{tt} - c^2 \nabla^2)\mathbf{A} = -\sum_i \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t)) \dot{\mathbf{x}}_i(t), \tag{2.12}$$

which closely resemble the equations for electrodynamics in the Lorentz gauge. (See, for example, Landau & Lifshitz (1975), p. 109.) The rhs of (2.11) and (2.12) correspond to charge density and to the two components of electric current density. These equations determine the response of the fields to the motion of the point vortices. Eqn. (2.9) determines the velocity of the point vortices in terms of the fields. In terms of the potentials defined by (2.2), (2.9) takes the form

$$(B_x - A_y)\dot{x}_i(t) = \phi_y - B_t, \tag{2.13}$$

$$(B_x - A_y)\dot{y}_i(t) = A_t - \phi_x. \tag{2.14}$$

The complete dynamics is described by (2.11)-(2.14).

To complete the analogy with classical electrodynamics, we seek a variational principle equivalent to (2.4), (2.7) and (2.13)-(2.14). (The variational principle should not refer to a particular gauge.) It is not difficult to guess that the Lagrangian takes the form

$$L[\phi(\mathbf{x}, t), \mathbf{A}(\mathbf{x}, t), \mathbf{x}_i(t)] = L_1[\phi, \mathbf{A}] + L_2[\phi, \mathbf{A}, \mathbf{x}_i], \tag{2.15}$$

where

$$L_1 = \tfrac{1}{2} \iiint dt\, d\mathbf{x} \left(c^2 (B_x - A_y)^2 - (A_t - \phi_x)^2 - (B_t - \phi_y)^2 \right) \tag{2.16}$$

depends only on the fields, and

$$L_2 = \iiint dt\, d\mathbf{x} \sum_i \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t)) \left(\phi(\mathbf{x}, t) + \mathbf{A}(\mathbf{x}, t) \cdot \dot{\mathbf{x}}_i(t)\right) \tag{2.17}$$

represents the coupling between the vortices and the fields. We obtain the field equations by requiring that $\delta L = 0$ for arbitrary variations $\delta\phi(\mathbf{x}, t)$ and $\delta\mathbf{A}(\mathbf{x}, t)$ in the fields, and we obtain the equations (2.13)-(2.14) for the evolution of the vortices by requiring that $\delta L = 0$ for arbitrary variations $\delta\mathbf{x}_i(t)$ in the locations of the vortices. To see the latter, we rewrite

$$L_2 = \sum_i \Gamma_i \int dt \left(\phi(\mathbf{x}_i(t), t) + A(\mathbf{x}_i(t), t)\dot{x}_i(t) + B(\mathbf{x}_i(t), t)\dot{y}_i(t)\right) \tag{2.18}$$

and consider variations $\delta x_i(t)$ in the x-component of the location of the i-th vortex. We find that

$$\delta L_2 = \int dt \left[(\phi_x + A_x \dot{x}_i(t) + B_x \dot{y}_i(t))\, \delta x_i(t) + A \frac{d}{dt} \delta x_i(t) \right]$$
$$= \int dt \left[\phi_x + A_x \dot{x}_i(t) + B_x \dot{y}_i(t) - \frac{d}{dt} A \right] \delta x_i(t)$$
$$= \int dt \left[\phi_x + B_x \dot{y}_i(t) - A_t - A_y \dot{y}_i(t) \right] \delta x_i(t). \tag{2.19}$$

Since $\delta L_2 = 0$ for arbitrary $\delta x_i(t)$, (2.14) follows. Similarly, $\delta L_2/\delta y_i(t) = 0$ implies (2.13).

Our variational principle bears a striking resemblance to the corresponding variational principle for classical electrodynamics (e.g. Landau & Lifshitz, 1975, p. 69). Our L_1 corresponds to the terms conventionally written as $F_{\mu\nu}F^{\mu\nu}$, and our L_2 corresponds to the terms conventionally written as $A_\mu j^\mu$. Moreover, there is a gauge symmetry corresponding to (2.3). Specifically, variations of the form $\delta\phi = (\delta\lambda)_t$ and $\delta\mathbf{A} = \nabla(\delta\lambda)$ do not affect L_1, but $\delta L_2/\delta\lambda = 0$ implies a conservation law for the vorticity (2.8). Thus the gauge symmetry of the WV Lagrangian corresponds to vorticity conservation in the same way that the gauge symmetry of classical electrodynamics corresponds to charge conservation.

The main difference between the variational principles for WV dynamics and electrodynamics is that the WV Lagrangian lacks a term that depends only on the vortex locations. That is, terms analogous to those that represent the inertia of charged particles are missing from our formulation. This corresponds to the fact that the point vortices move at a *velocity* that is determined by the fields and not, as in electrodynamics, with an *acceleration* determined by the fields.

We note that the solutions of (2.11)-(2.12) are singular at the locations of the point vortices. Thus, in applying (2.9), we must omit the contribution of the i-th point vortex to the fields that appear on the rhs. This restriction, which also occurs in conventional point vortex dynamics, is analogous to neglecting the force of charged particles on themselves. It has (I believe) no fundamental justification.

If we neglect the ∂_{tt}-terms in (2.11)-(2.12), then the resulting equations are elliptic. Taking the solutions of these elliptic equations as the leading order terms in an 'action-at-a-distance' approximation, we obtain $\phi(\mathbf{x},t) = \phi_0(\mathbf{x},t) + O(c^{-2})$, where

$$\phi_0(\mathbf{x},t) = -\sum_i \frac{\Gamma_i}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_i(t)|), \tag{2.20}$$

and $\mathbf{A}(\mathbf{x},t) = (0,x) + O(c^{-2})$, where we have been careful to add the homogeneous solution $B = x$ required to satisfy the boundary condition $\alpha = B_x - A_y \to 1$ as $r \to \infty$. Substituting these approximations back into (2.16) and (2.17), and neglecting irrelevant constants and terms of order c^{-2}, we obtain

$$L_1 = \tfrac{1}{2} \iiint dt\, d\mathbf{x}\, (-\nabla\phi_0 \cdot \nabla\phi_0)) = \tfrac{1}{2} \int dt \sum_i \sum_j \frac{\Gamma_i\Gamma_j}{2\pi} \ln r_{ij}(t) \tag{2.21}$$

and

$$L_2 = \int dt \sum_i \Gamma_i \left(\phi_0(\mathbf{x}_i(t)) + x_i(t)\dot{y}_i(t) \right))$$
$$= \int dt \sum_i \Gamma_i x_i(t) \dot{y}_i(t) - \int dt \sum_i \sum_j \frac{\Gamma_i\Gamma_j}{2\pi} \ln r_{ij}(t), \tag{2.22}$$

where $r_{ij}(t) = |\mathbf{x}_i(t) - \mathbf{x}_j(t)|$. The resulting

$$L = L_1 + L_2 = \int dt \sum_i \Gamma_i x_i(t)\dot{y}_i(t) - \sum_i \sum_{j>i} \frac{\Gamma_i \Gamma_j}{2\pi} \int dt \ \ln r_{ij}(t) \tag{2.23}$$

is the Lagrangian for conventional point vortex dynamics, which we henceforth call *Kirchhoff dynamics*, after one of its originators. For a review of Kirchhoff dynamics, see Aref (2007). In writing out the sums in (2.23), we have thrown away the self-interaction corresponding to $i = j$.

3. Fields generated by a point vortex with a prescribed trajectory

Now we consider a single point vortex with prescribed trajectory $\mathbf{x}_1(t)$ on the infinite plane. We adopt the gauge condition (2.5) so that the fields are determined by (2.11)-(2.12). These equations take the form of two-dimensional wave equations with a prescribed source term. However, it is convenient to consider them as special cases of the three-dimensional wave equation

$$(\partial_{tt} - c^2(\partial_{xx} + \partial_{yy} + \partial_{zz}))f(x, y, z, t) = Q(x, y, z, t), \tag{3.1}$$

in which Q and hence f is independent of z. The point vortices are lines parallel to the z-axis. The causal solution of (3.1) is

$$f(\mathbf{X}, t) = \frac{1}{4\pi c^2} \iiiint d\mathbf{X}_0 dt_0 \ \frac{Q(\mathbf{X}_0, t_0)}{|\mathbf{X} - \mathbf{X}_0|} \delta(t_0 - t_r), \tag{3.2}$$

where

$$t_r = t - |\mathbf{X} - \mathbf{X}_0|/c \tag{3.3}$$

is the 'retarded time.' Our notation is $\mathbf{X} = (\mathbf{x}, z) = (x, y, z)$. With no loss in generality we may set $z = 0$ and write (3.2) as

$$f(\mathbf{x}, t) = \frac{1}{4\pi c^2} \iiiint d\mathbf{x}_0 dZ_0 dt_0 \ \frac{Q(\mathbf{x}_0, t_0)}{|\mathbf{X} - \mathbf{X}_0|} \delta(t_0 - t_r). \tag{3.4}$$

Then by (2.11)-(2.12) we have

$$(\phi(\mathbf{x}, t), A(\mathbf{x}, t), B(\mathbf{x}, t)) = \frac{1}{4\pi c^2} \iiint d\mathbf{x}_0 dz \ \frac{\Gamma_1(c^2, \ -\dot{x}_1(t_r), \ -\dot{y}_1(t_r))}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}} \delta(\mathbf{x}_0 - \mathbf{x}_1(t_r)) \tag{3.5}$$

where now

$$t_r = t - \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}/c, \tag{3.6}$$

and we have replaced the dummy integration variable Z_0 by z. Next we use the delta functions to eliminate the integration over x_0 and y_0. This requires a transformation of the integration variables from $\mathbf{x}_0$ to $\mathbf{x}_0 - \mathbf{x}_1(t_r)$. The Jacobian of transformation is

$$\frac{\partial(\mathbf{x}_0 - \mathbf{x}_1(t_r))}{\partial(\mathbf{x}_0)} = 1 - \frac{\dot{\mathbf{x}}_1(t_r) \cdot (\mathbf{x} - \mathbf{x}_0)}{c\sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}}. \tag{3.7}$$

Thus (3.5) becomes

$$(\phi(\mathbf{x}, t), A(\mathbf{x}, t), B(\mathbf{x}, t)) = \frac{\Gamma_1}{4\pi c^2} \int_{-\infty}^{+\infty} dz \ \frac{(c^2, \ -\dot{x}_1(t_r), \ -\dot{y}_1(t_r))}{D_1}, \tag{3.8}$$

where

$$D_1 = \sqrt{r_1^2 + z^2} - \dot{\mathbf{x}}_1(t_1) \cdot \mathbf{r}_1/c, \tag{3.9}$$

$$t_1 = t - \sqrt{r_1^2 + z^2}/c, \tag{3.10}$$

$$\mathbf{r}_1 = \mathbf{x} - \mathbf{x}_1(t_1), \tag{3.11}$$

and $r_1 = |\mathbf{r}_1|$. Eqns (3.10) and (3.11) implicitly define $\mathbf{r}_1(x, y, z, t)$, $t_1(x, y, z, t)$, and hence $D_1(x, y, z, t)$. To the solutions (3.8) we must be prepared to add homogeneous solutions of (2.11)-(2.12) as needed to satisfy boundary conditions.

Our interest is in the 'physical fields' (α, u, v) and these require derivatives of (ϕ, A, B) with respect to x, y and t. For example, $\alpha = B_x - A_y$, where

$$\mathbf{A}(\mathbf{x}, t) = (0, x) - \frac{\Gamma_1}{4\pi c^2} \int_{-\infty}^{+\infty} dz\, \frac{\dot{\mathbf{x}}_1(t_r)}{D_1}, \tag{3.12}$$

and we have added a homogeneous solution needed to satisfy the boundary condition that $\alpha \to 1$ as $|\mathbf{x}| \to \infty$. To evaluate

$$\alpha(\mathbf{x}, t) = 1 + \frac{\Gamma_1}{4\pi c^2} \int_{-\infty}^{+\infty} dz \left(\frac{\partial}{\partial y} \frac{\dot{x}_1(t_1)}{D_1} - \frac{\partial}{\partial x} \frac{\dot{y}_1(t_1)}{D_1} \right) \tag{3.13}$$

we must compute ∇D_1 and ∇t_1. From the definitions (3.10)-(3.11) we obtain

$$\nabla r_1 = -\frac{\sqrt{r_1^2 + z^2}}{r_1} c \nabla t_1, \tag{3.14}$$

$$c\nabla t_1 = -\mathbf{r}_1/D_1. \tag{3.15}$$

Using these results to compute the gradient of D_1 we obtain

$$\nabla D_1 = -\frac{\mathbf{v}_1}{c} + \frac{1}{D_1} \left(1 + \frac{\mathbf{a}_1 \cdot \mathbf{r}_1}{c^2} - \frac{\mathbf{v}_1 \cdot \mathbf{v}_1}{c^2} \right) \mathbf{r}_1, \tag{3.16}$$

where we have adopted the shorthand notation $\mathbf{v}_1 \equiv \dot{\mathbf{x}}_1(t_1)$ and $\mathbf{a}_1 \equiv \ddot{\mathbf{x}}_1(t_1)$. Then (3.13) becomes

$$\alpha(\mathbf{x}, t) = 1 + \frac{\Gamma_1}{4\pi c^2} \int_{-\infty}^{+\infty} dz \left(\frac{1}{c{D_1}^2} (\mathbf{r}_1 \times \mathbf{a}_1) + \frac{1}{{D_1}^3} \left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_1}{c^2} + \frac{\mathbf{a}_1 \cdot \mathbf{r}_1}{c^2} \right) (\mathbf{r}_1 \times \mathbf{v}_1) \right). \tag{3.17}$$

Since the equations (2.11)-(2.12) determining the fields are linear equations, we obtain the pressure field corresponding to N point vortices by superposition: Change all the 1-subscripts above to i-subscripts and sum the integral term in (3.17) from $i = 1$ to N. The z-integration in (3.17) corresponds to an integration back in time, as the signal from larger $|z|$ requires a greater time to arrive at $z = 0$. The acceleration terms (i.e. the $\mathbf{a}_1$-terms) in (3.17) radiate energy away to infinity; the remaining terms represent the pressure field carried along by the point vortex. Thus, as in electrodynamics, only accelerating charges (vortices) radiate. The $\mathbf{r}_1 \times \mathbf{a}_1$ term is responsible for dipole radiation. The $\mathbf{a}_1 \cdot \mathbf{r}_1$ term is responsible for quadrupole radiation. We give several examples to illustrate these points.

As a first example, let

$$\mathbf{x}_1(t) = (V_0 t, 0) \tag{3.18}$$

be the location of a single vortex moving along the x-axis at the constant prescribed speed V_0. Then $\mathbf{v}_1 = (V_0, 0)$, $\mathbf{a}_1 = 0$, and (3.17) reduces to

$$\alpha(\mathbf{x}, t) = 1 - \frac{y V_0 \Gamma_1}{4\pi c^2} (1 - \epsilon^2) \int_{-\infty}^{+\infty} dz\, \frac{1}{{D_1}^3}, \tag{3.19}$$

where $\epsilon = V_0/c$ is the Froude number. Solving (3.9)-(3.11) for

$$D_1{}^2 = (x - V_0 t)^2 + (1 - \epsilon^2)(y^2 + z^2), \tag{3.20}$$

putting (3.20) into (3.19), and carrying out the integration we finally obtain

$$\alpha(\mathbf{x}, t) = 1 - \frac{V_0 \Gamma_1}{2\pi c^2} \frac{\sqrt{1-\epsilon^2}\; y}{(x - V_0 t)^2 + (1-\epsilon^2) y^2} \tag{3.21}$$

and

$$\mathbf{u}(\mathbf{x}, t) = -c^2 \int_{-\infty}^{t} dt' \, \nabla \alpha(\mathbf{x}, t') = \frac{\Gamma_1 \sqrt{1-\epsilon^2}}{2\pi} \frac{(-y, \; x - V_0 t)}{(x - V_0 t)^2 + (1-\epsilon^2) y^2}. \tag{3.22}$$

The results (3.21) and (3.22) are exact within the context of WV dynamics. As $\epsilon \to 0$, $\alpha \to 1$ and the velocity (3.22) approaches that of Kirchhoff dynamics. For $\epsilon \neq 0$ the velocity field (3.22) is divergent. We find that

$$\nabla \cdot \mathbf{u} = \frac{\epsilon^2 \Gamma_1}{\pi} \frac{(x - V_0 t)\sqrt{1-\epsilon^2}\; y}{((x - V_0 t)^2 + (1-\epsilon^2) y^2)^2}. \tag{3.23}$$

Of course, a single vortex cannot move by itself; for the solution (3.21)-(3.22) to have meaning we must imagine that the vortex is being pushed along by an outside force. However, we obtain a consistent solution by admitting a second vortex with $\Gamma_2 = -\Gamma_1$ located on $x = x_1$ at the point where $u_1/\alpha_1 = V_0$ and remembering that neither vortex experiences its own field. This two-vortex solution is a generalization of the corresponding solution for counter-rotating vortices in Kirchhoff dynamics.

As a second example, we consider the pressure field generated by a point vortex oscillating along the x-axis,

$$\mathbf{x}_1(t) = (\frac{V_0}{\omega} \sin \omega t, \; 0). \tag{3.24}$$

In this case it is impossible to evaluate (3.17) exactly. We obtain the solution for $r = |\mathbf{x}| \to \infty$ by the method of stationary phase. The details of this calculation are standard, and we present only the result. To the leading order in r^{-1} and c^{-1}, only the $\mathbf{r}_1 \times \mathbf{a}_1$ term in (3.17) contributes, and we find that

$$\alpha(\mathbf{x}, t) \sim 1 + \frac{\epsilon \omega \Gamma_1}{4\pi c^2} \sqrt{\frac{2\pi}{Kr}} \sin\theta \sin(\omega t + \pi/4), \tag{3.25}$$

where $K = \omega/c$ is the wavenumber associated with the oscillation and $\theta = \tan^{-1} y/x$. The solution (3.25) corresponds to dipole radiation at right angles to the trajectory of the vortex. Again, it is impossible for a vortex to oscillate in such a manner by itself, but we may consider (3.25) as the Born approximation for the response of a vortex to an incoming wave.

As a final example, we consider a pair of co-rotating vortices with $\mathbf{x}_1(t) = -\mathbf{x}_2(t)$ and equal strength Γ. We assume that the vortices rotate in a circle of radius r_0 at velocity $V_0 = \Gamma/4\pi r_0$ and angular speed $\Omega = \Gamma/4\pi {r_0}^2$. This is the Kirchoff approximation to the near field; by making it we escape the difficult task of solving (2.11)-(2.14) as a coupled set. This calculation is very similar to that of Powell (1964); see Howe (2003, p. 120). Again we are interested in the solution as $r \to \infty$, and we present only the final result. To leading order in r^{-1} and c^{-1} the contributions of the vortices to the dipole term in (3.17) cancel, and only the $\mathbf{a}_i \cdot \mathbf{r}_i$ terms contribute. We find that

$$\alpha(r, \theta) \sim 1 + \frac{\Gamma V_0^2 \Omega}{4\pi c^4} \sqrt{\frac{\pi}{Kr}} \cos(2Kr - 2\Omega t - 2\theta), \tag{3.26}$$

where $K = \Omega/c$, corresponding to a weak quadrupole radiation. In the exact solution of the problem, the two co-rotating vortices would gradually move apart at a rate determined by the radiative loss of energy.

4. Remarks

WV dynamics certainly omit phenomena of physical importance. Wave-breaking and shock formation are only the most obvious of these. Howe (1999) considers the scattering of an acoustic wave by a point vortex in the Born approximation, using dynamics analogous to (1.1)-(1.2), in which wave refraction can occur in regions of vanishing vorticity. He finds that the scattered wave includes a dipole radiation term identical to (3.25). However, there is an additional, equally important, scattering contribution from the flow circulating around the point vortex. Thus the property of WV dynamics that the waves interact only with vorticity can lead to inaccurate results.

On the other hand, one could hardly expect to produce a close analogy between fluid mechanics and classical electrodynamics without making a strong approximation. Electromagnetic waves do not 'break'; they interact only with charged particles. Hence electrodynamics and fluid mechanics can never be made equivalent. However, there are situations in which the unrealistic properties of WV dynamics may be desirable. In a study of the interaction between waves and mean flows using the shallow-water equations, Bühler (1998) found it convenient to remove shock formation from the dynamics by an artificial modification of the potential energy in (1.5). WV dynamics achieves the same end by a modifying the kinetic energy to that in (1.6).

WV dynamics may be a useful new starting point for applying renormalised perturbation theories like Kraichnan's (1959) direct interaction approximation to the study of two-dimensional turbulence. In their usual formulation, these techniques apply to any field theory with quadratic nonlinearity and thus ignore the special status of vorticity in fluid mechanics. Similar methods applied to electrodynamics pay great attention to electric charge.

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