

Entropy budget and coherent structures associated with a spectral closure model of turbulence

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We ‘derive’ the eddy-damped quasi-normal Markovian model (EDQNM) by a method that replaces the exact equation for the Fourier phases with a solvable stochastic model, and we analyse the entropy budget of EDQNM. We show that a quantity that appears in the probability distribution of the phases may be interpreted as the rate at which entropy is transferred from the Fourier phases to the Fourier amplitudes. In this interpretation, the decrease in phase entropy is associated with the formation of structures in the flow, and the increase of amplitude entropy is associated with the spreading of the energy spectrum in wavenumber space. We use Monte Carlo methods to sample the probability distribution of the phases predicted by our theory. This distribution contains a single adjustable parameter that corresponds to the triad correlation time in EDQNM. Flow structures form as the triad correlation time becomes very large, but the structures take the form of vorticity quadrupoles that do not resemble the monopoles and dipoles that are actually observed.

1. Introduction

In the ‘standard model’ of two-dimensional turbulence, due mainly to Kraichnan (1967), Batchelor (1969), and Leith (1968) (hereafter KBL), forcing at an intermediate wavenumber produces a leftward (to lower wavenumber) cascade of energy in a $k^{-5/3}$ inertial range, and a rightward enstrophy cascade in a k^{-3} inertial range. Simple and compelling arguments predict the existence of these ranges, but nothing in the standard model anticipates the isolated coherent vortices discovered by McWilliams (1984). Although numerical experiments show these vortices to be well defined and ubiquitous, there is as yet no compelling theoretical explanation for their existence: If they had never been observed, no one would be greatly surprised. However, Benzi et al (1988) suggest that the vortex population statistics obey a self-similar scaling that resembles the KBL scaling of the inertial ranges. This self-similarity, which is supported by the work of Burgess et al (2017), encourages the hope that the coherent vortices might yet fit within the standard model of two-dimensional turbulence. For a recent review of two-dimensional turbulence see Boffetta & Ecke (2012).

Turbulence closure models of the direct-interaction family provide a quantitative theoretical foundation for the inertial range theory of two-dimensional turbulence, but it is generally agreed that these models have nothing to say about the formation of structures in the flow: The closure models predict the evolution of the Fourier amplitudes, whereas the structures clearly depend on the phases of the Fourier coefficients. However, it is possible to ‘derive’ a well known spectral closure model—the eddy-damped quasi-normal Markovian model (hereafter EDQNM)—by a method that exposes the closure hypoth-

esis as an assumption about the phases of the Fourier coefficients. By examining this hypothesis on phases, we investigate flow structures that are consistent with EDQNM. For an introduction to EDQNM see Orszag (1970) and Lesieur (1987).

The plan of the paper is as follows. In Section 2 we ‘derive’ EDQNM by a method that replaces the exact equation for the Fourier phases with a solvable stochastic model. This method of derivation demonstrates the primary importance of the principle of entropy increase in EDQNM. Section 3 analyses the entropy budget of EDQNM. We show that a quantity that appears in the probability distribution of the phases may be interpreted as the rate at which entropy is transferred from the Fourier phases to the Fourier amplitudes. In this interpretation, the decrease in phase entropy is associated with the formation of structures in the flow, and the increase of amplitude entropy is associated with the spreading of the energy spectrum in wavenumber space. In Section 4 we use Monte Carlo methods to sample the probability distribution of the phases predicted by our theory. This distribution contains a single adjustable parameter that corresponds to the triad correlation time in EDQNM. Flow structures form as the triad correlation time becomes very large, but the structures take the form of vorticity quadrupoles that do not resemble the monopoles and dipoles that are actually observed. Section 5 concludes with an assessment of our results.

2. EDQNM

We consider freely decaying, two-dimensional turbulence governed by

$$\zeta_t + \mathbf{v} \cdot \nabla \zeta = \nu \nabla^2 \zeta, \quad (2.1)$$

where $\mathbf{v} = (-\psi_y, \psi_x)$ is the fluid velocity and $\zeta = \nabla^2 \psi$ is the vorticity. The flow is 2π -periodic in x and y . We introduce the Fourier representation

$$\psi(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\psi}_{\mathbf{k}}(t) e^{i(\mathbf{k} \cdot \mathbf{x})}, \quad (2.2)$$

where $\mathbf{x} = (x, y)$ and the sum is over integer pairs $\mathbf{k} = (k_x, k_y)$ in the wavenumber plane. We let

$$\hat{\psi}_{\mathbf{k}} = A_{\mathbf{k}} \exp(i\phi_{\mathbf{k}}), \quad (2.3)$$

where the amplitude $A_{\mathbf{k}}$ is real and positive, and the phase $\phi_{\mathbf{k}}$ is real. Since ψ is real, $A_{-\mathbf{k}} = A_{\mathbf{k}}$ and $\phi_{-\mathbf{k}} = -\phi_{\mathbf{k}}$. The Fourier transform of (2.1) is

$$\frac{d}{dt} A_{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{p}} \sum_{\mathbf{q}} (\mathbf{k} \times \mathbf{p}) \frac{q^2 - p^2}{k^2} A_{\mathbf{p}} A_{\mathbf{q}} \cos(\phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}}) \delta_{\mathbf{k}+\mathbf{p}+\mathbf{q}} - \nu k^2 A_{\mathbf{k}} \quad (2.4)$$

and

$$\frac{d}{dt} \phi_{\mathbf{k}} = \frac{1}{2A_{\mathbf{k}}} \sum_{\mathbf{p}} \sum_{\mathbf{q}} (\mathbf{k} \times \mathbf{p}) \frac{p^2 - q^2}{k^2} A_{\mathbf{p}} A_{\mathbf{q}} \sin(\phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}}) \delta_{\mathbf{k}+\mathbf{p}+\mathbf{q}}, \quad (2.5)$$

where $k = |\mathbf{k}|$.

To obtain EDQNM we regard the $A_{\mathbf{k}}$ as definite (i.e. statistically sharp) and the $\phi_{\mathbf{k}}$ as random. Then the average of (2.4) is

$$\frac{d}{dt} A_{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{p}} \sum_{\mathbf{q}} (\mathbf{k} \times \mathbf{p}) \frac{q^2 - p^2}{k^2} A_{\mathbf{p}} A_{\mathbf{q}} \langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle \delta_{\mathbf{k}+\mathbf{p}+\mathbf{q}} - \nu k^2 A_{\mathbf{k}}, \quad (2.6)$$

where

$$\xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \equiv \phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}} \quad (2.7)$$

and $\langle \rangle$ denotes the average. Closure at the level of the spectrum requires that $\langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle$ be replaced by an approximation that involves only the amplitudes. To this end, we use (2.5) to write the evolution equation,

$$\begin{aligned} \frac{d}{dt} \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} &= \frac{d}{dt} (\phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}}) \\ &= \frac{1}{2A_{\mathbf{k}}} \sum_{\mathbf{r}} \sum_{\mathbf{s}} (\mathbf{k} \times \mathbf{r}) \frac{r^2 - s^2}{k^2} A_{\mathbf{r}} A_{\mathbf{s}} \sin(\xi_{\mathbf{k}\mathbf{r}\mathbf{s}}) \delta_{\mathbf{k}+\mathbf{r}+\mathbf{s}} \\ &\quad + \frac{1}{2A_{\mathbf{p}}} \sum_{\mathbf{r}} \sum_{\mathbf{s}} (\mathbf{p} \times \mathbf{r}) \frac{r^2 - s^2}{p^2} A_{\mathbf{r}} A_{\mathbf{s}} \sin(\xi_{\mathbf{p}\mathbf{r}\mathbf{s}}) \delta_{\mathbf{p}+\mathbf{r}+\mathbf{s}} \\ &\quad + \frac{1}{2A_{\mathbf{q}}} \sum_{\mathbf{r}} \sum_{\mathbf{s}} (\mathbf{q} \times \mathbf{r}) \frac{r^2 - s^2}{q^2} A_{\mathbf{r}} A_{\mathbf{s}} \sin(\xi_{\mathbf{q}\mathbf{r}\mathbf{s}}) \delta_{\mathbf{q}+\mathbf{r}+\mathbf{s}}, \end{aligned} \quad (2.8)$$

for $\xi_{\mathbf{k}\mathbf{p}\mathbf{q}}$. Next we rewrite the right hand side of (2.8) as the sum of ‘direct interaction’ terms, in which \mathbf{r} and \mathbf{s} are equal to \mathbf{k} , \mathbf{p} , or \mathbf{q} , and a (generally much larger) remainder term that includes all the other values of \mathbf{r} and \mathbf{s} . Thus,

$$\frac{d}{dt} \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} = B_{\mathbf{k}\mathbf{p}\mathbf{q}} \sin \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} + R_{\mathbf{k}\mathbf{p}\mathbf{q}}, \quad (2.9)$$

where

$$B_{\mathbf{k}\mathbf{p}\mathbf{q}} = (\mathbf{k} \times \mathbf{p}) \left(\frac{q^2 - k^2}{p^2} \frac{A_{\mathbf{k}} A_{\mathbf{q}}}{A_{\mathbf{p}}} + \frac{k^2 - p^2}{q^2} \frac{A_{\mathbf{k}} A_{\mathbf{p}}}{A_{\mathbf{q}}} + \frac{p^2 - q^2}{k^2} \frac{A_{\mathbf{p}} A_{\mathbf{q}}}{A_{\mathbf{k}}} \right) \delta_{\mathbf{k}+\mathbf{p}+\mathbf{q}} \quad (2.10)$$

and $R_{\mathbf{k}\mathbf{p}\mathbf{q}}$ is the remainder. Finally, we set

$$R_{\mathbf{k}\mathbf{p}\mathbf{q}} = W_{\mathbf{k}\mathbf{p}\mathbf{q}}(t), \quad (2.11)$$

where $W_{\mathbf{k}\mathbf{p}\mathbf{q}}(t)$ is a white noise process with a prescribed covariance,

$$\langle W_{\mathbf{k}\mathbf{p}\mathbf{q}}(t) W_{\mathbf{k}\mathbf{p}\mathbf{q}}(t') \rangle = 2D_{\mathbf{k}\mathbf{p}\mathbf{q}} \delta(t - t'). \quad (2.12)$$

Note that $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ and $\xi_{\mathbf{k}\mathbf{p}\mathbf{q}}$ are invariant to permutations in their vector subscripts. The same must therefore be true of $R_{\mathbf{k}\mathbf{p}\mathbf{q}}$, $W_{\mathbf{k}\mathbf{p}\mathbf{q}}$, and $D_{\mathbf{k}\mathbf{p}\mathbf{q}}$.

Suppose that the $D_{\mathbf{k}\mathbf{p}\mathbf{q}}$ have been chosen. Let $P_{\mathbf{k},\mathbf{p},\mathbf{q}}(\xi, t)$ be the probability distribution of $\xi_{\mathbf{k}\mathbf{p}\mathbf{q}}$. Then, temporarily omitting the vector subscripts to ease the notation, we find that $P(\xi, t)$ obeys the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial \xi} (B \sin \xi \cdot P) = D \frac{\partial^2 P}{\partial \xi^2}. \quad (2.13)$$

This is a separate equation for every triad. In statistically steady or slowly evolving flow, the time derivative is negligible, and the solution, which must be periodic in ξ , is

$$P(\xi) = C \exp(-B \cos \xi / D), \quad (2.14)$$

where C is the normalization constant. From (2.14) it follows that

$$\langle \cos \xi \rangle = -\frac{I_1(B/D)}{I_0(B/D)}, \quad (2.15)$$

where I_0 and I_1 are modified Bessel functions. If B/D is small, then

$$P(\xi) \approx (1 - B \cos \xi / D) / 2\pi \quad (2.16)$$

and, restoring the vector subscripts,

$$\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle \approx -\frac{1}{2} \frac{B_{\mathbf{k}\mathbf{p}\mathbf{q}}}{D_{\mathbf{k}\mathbf{p}\mathbf{q}}}. \quad (2.17)$$

Let $U_{\mathbf{k}} = \frac{1}{2} k^2 A_{\mathbf{k}}^2$ be the energy in mode \mathbf{k} . Multiplying (2.6) by $k^2 A_{\mathbf{k}}$ and using (2.10) and (2.17), we obtain the spectral evolution equation

$$\begin{aligned} \frac{d}{dt} U_{\mathbf{k}} = & \sum_{\mathbf{p}} \sum_{\mathbf{q}} \theta_{\mathbf{k}\mathbf{p}\mathbf{q}} \frac{(\mathbf{k} \times \mathbf{p})^2}{k^2 p^2 q^2} [(q^2 - p^2)^2 U_{\mathbf{p}} U_{\mathbf{q}} - 2(q^2 - p^2)(q^2 - k^2) U_{\mathbf{k}} U_{\mathbf{q}}] \delta_{\mathbf{k}+\mathbf{p}+\mathbf{q}} \\ & - 2\nu k^2 U_{\mathbf{k}}, \end{aligned} \quad (2.18)$$

where

$$\theta_{\mathbf{k}\mathbf{p}\mathbf{q}} \equiv \frac{1}{D_{\mathbf{k}\mathbf{p}\mathbf{q}}}. \quad (2.19)$$

Eq. (2.18) is the standard form of the EDQNM. In the usual interpretation, $\theta_{\mathbf{k}\mathbf{p}\mathbf{q}}$, which is symmetric with respect to permutations of its vector subscripts, is the average time over which the phases corresponding to wavenumbers \mathbf{k} , \mathbf{p} , \mathbf{q} remain correlated. The $\theta_{\mathbf{k}\mathbf{p}\mathbf{q}}$ are often considered to be free parameters of the theory. A typical choice is

$$D_{\mathbf{k}\mathbf{p}\mathbf{q}} = \mu(\mathbf{k}) + \mu(\mathbf{p}) + \mu(\mathbf{q}), \quad (2.20)$$

where

$$\mu(\mathbf{k}) = g \sum_{|\mathbf{p}| < |\mathbf{k}|} p^2 U_{\mathbf{p}} \quad (2.21)$$

is proportional to the strain in scales larger than k^{-1} , and g is an order-one dimensionless constant. The choice (2.20)-(2.21) is consistent with Kolmogorov theory, and the constant g may be adjusted to agree with measured values of Kolmogorov's constant in either inertial range. At the two extremes, Kraichnan's (1971) test field model offers a systematically derived expression for $\theta_{\mathbf{k}\mathbf{p}\mathbf{q}}$, while Frisch et al (1974) simply take $\theta_{\mathbf{k}\mathbf{p}\mathbf{q}}$ to be a constant (independent of \mathbf{k} , \mathbf{p} , \mathbf{q}). Experience shows that solutions of (2.18) are relatively insensitive to the precise choice of $\theta_{\mathbf{k}\mathbf{p}\mathbf{q}}$.

Eqn. (2.17) is the critical assumption that removes all of the phase information to yield a closed evolution equation (2.18) in terms of the spectral amplitudes alone. The validity of (2.17) rests upon the validity of (2.18), which has proved itself in many applications. However, if (2.17) is valid, then the information that it contains about the phases may also have value. That is, (2.17) may contain information about the structure of the flow field. In the remainder of this paper we explore this possibility. But first we recall some important properties of EDQNM.

When $\nu = 0$, (2.18) conserves all the quantities that are conserved by the exact dynamics and can be expressed solely in terms of the amplitudes $A_{\mathbf{k}}$. These include the energy,

$$E = \sum_{\mathbf{k}} U_{\mathbf{k}}, \quad (2.22)$$

and the enstrophy

$$Z = \sum_{\mathbf{k}} k^2 U_{\mathbf{k}}. \quad (2.23)$$

This may be shown directly from (2.18), but it is immediate from (2.6), which holds for arbitrary values of the amplitudes and phases on its right hand side (arbitrary initial conditions), and thus for arbitrary $\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle$. Thus the choice (2.17) that corresponds

to EDQNM is not determined by the need to maintain conservation laws. Instead, as we shall see, it is more closely associated with the principle of entropy increase.

Carnevale et al (1981) showed that when $\nu = 0$, (2.18) satisfies an H-theorem. In our notation,

$$\frac{d}{dt} \sum_{\mathbf{k}} 2 \ln A_{\mathbf{k}} > 0. \quad (2.24)$$

Here we offer a brief justification of (2.24). In the following section we give a more thorough discussion of entropy and its evolution in EDQNM.

The reasoning behind (2.24) runs as follows. First, it is easy to show that when $\nu = 0$ the motion in the phase space spanned by the real and imaginary parts of $\psi_{\mathbf{k}}$ is nondivergent. That is, Liouville's theorem applies. The variables $A_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ represent polar coordinates in the subspace corresponding to mode \mathbf{k} . The entropy associated with a knowledge of the amplitudes *alone* is proportional to the logarithm of the volume in phase space that corresponds to the set of amplitude values $\{A_{\mathbf{k}}\}$. For each $A_{\mathbf{k}}$, this volume corresponds to a circle of radius $A_{\mathbf{k}}$ in the subspace corresponding to mode \mathbf{k} . Assuming no knowledge of the $\phi_{\mathbf{k}}$'s, each point on this circle is equally probable. Since the circumference of the circle is proportional to its radius, the entropy must be proportional to

$$S = \sum_{\mathbf{k}} 2 \ln A_{\mathbf{k}} = \sum_{\mathbf{k}} \ln U_{\mathbf{k}}. \quad (2.25)$$

The principle of mixing in phase space dictates that the amplitude values successively predicted by EDQNM must correspond to successively larger values of the entropy; mixing in phase space can only degrade information. Thus EDQNM must obey $dS/dt > 0$. We check this by direct calculation. First we rewrite (2.24) as

$$\frac{dS}{dt} = \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} 2 \left(\frac{\dot{A}_{\mathbf{k}}}{A_{\mathbf{k}}} + \frac{\dot{A}_{\mathbf{p}}}{A_{\mathbf{p}}} + \frac{\dot{A}_{\mathbf{q}}}{A_{\mathbf{q}}} \right), \quad (2.26)$$

where the sum is over all the triads in the system, and the overdots denote the rate of change due to the other two members of the triad. Since the initial conditions are arbitrary and may be such that only a single triad is excited, each triad must make a positive contribution to (2.26). That is, the summand in (2.26) must be positive. (We do not count permutations as separate triads. Thus if $[\mathbf{k}\mathbf{p}\mathbf{q}]$ occurs in the sum, then $[\mathbf{p}\mathbf{k}\mathbf{q}]$ does not.) By (2.6) the entropy increase due to triad $[\mathbf{k}\mathbf{p}\mathbf{q}]$ is

$$\begin{aligned} 2 \left(\frac{\dot{A}_{\mathbf{k}}}{A_{\mathbf{k}}} + \frac{\dot{A}_{\mathbf{p}}}{A_{\mathbf{p}}} + \frac{\dot{A}_{\mathbf{q}}}{A_{\mathbf{q}}} \right) &= (\mathbf{k} \times \mathbf{p}) \frac{q^2 - p^2}{k^2} \frac{A_{\mathbf{p}} A_{\mathbf{q}}}{A_{\mathbf{k}}} \langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle + \text{cyc}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \\ &= -B_{\mathbf{k}\mathbf{p}\mathbf{q}} \langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle, \end{aligned} \quad (2.27)$$

where $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ is given by (2.10). Thus

$$\frac{dS}{dt} = - \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} B_{\mathbf{k}\mathbf{p}\mathbf{q}} \langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle. \quad (2.28)$$

Eq. (2.28) holds for (2.6) in general. For the EDQNM closure hypothesis (2.17), we obtain

$$\frac{dS}{dt} = \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} \frac{1}{2} \frac{B_{\mathbf{k}\mathbf{p}\mathbf{q}}^2}{D_{\mathbf{k}\mathbf{p}\mathbf{q}}} > 0. \quad (2.29)$$

Salmon (1998) argues that the H-theorem associated with EDQNM is not merely an

incidental property of the theory, but rather that it, along with the conservation laws for energy and enstrophy, “virtually determines” the form that the theory can take. From (2.28) we see that the EDQNM closure hypothesis is not the only hypothesis that satisfies (2.24). The more general hypothesis

$$\langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle \propto - \left(\frac{B_{\mathbf{k}\mathbf{p}\mathbf{q}}}{D_{\mathbf{k}\mathbf{p}\mathbf{q}}} \right)^{2n+1} \quad (2.30)$$

satisfies the conservation and entropy properties for any integer n . The EDQNM hypothesis (2.17) corresponds to $n = 0$. The hypothesis (2.15), which does not assume small B/D , also satisfies (2.24), and in fact (2.30) are just the terms that appear in an expansion of (2.15) in powers of B/D .

If the viscosity is switched off, and if the dynamics (2.1) is truncated to a finite number of modes (typically $k < k_c$ for some cutoff k_c), then the system evolves to a statistically steady state that maximizes (2.25) subject to the constraints (2.22) and (2.23). This easy variational problem leads to the ‘absolute equilibrium’ state discovered by Kraichnan (1967), namely

$$U_{\mathbf{k}} = \frac{1}{\alpha + \beta k^2}, \quad (2.31)$$

where α and β are determined by (2.22) and (2.23) and the prescribed values of E and Z .

Carnevale (1982) tested these ideas in direct numerical simulations of inviscid, spectrally truncated, two-dimensional turbulence. He found that the entropy (2.25) increases monotonically, asymptotically approaching the maximum entropy state corresponding to (2.31). It is noteworthy that his computations of entropy involve no averaging of any kind: the $U_{\mathbf{k}}$ were calculated from a single numerical simulation at a single time. The addition of the many logarithms evidently cancels the statistical fluctuations of the terms.

In this section, we have adopted the viewpoint that EDQNM is solely concerned with the Fourier amplitudes, and that the entropy measures our complete ignorance of the phases. However, it is possible to regard the stochastic model (2.9) underlying EDQNM as a model of both the amplitudes and the phases. In this expanded view, entropy measures our combined ignorance of both. In the following section we adopt this second point of view. However, one important conclusion is already apparent. A complete lack of information about the phases corresponds to independent, uniformly distributed $\phi_{\mathbf{k}}$. (Such distribution also corresponds to the ‘random initial conditions’ that are commonly used in numerical simulations of freely decaying turbulence.) Uniformly distributed phases correspond to uniformly distributed $\xi_{\mathbf{k},\mathbf{p},\mathbf{q}}$ and hence to $\langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle = 0$. By (2.6) this shuts off the evolution of the $A_{\mathbf{k}}$ and the associated increase in the entropy (2.25) associated with the amplitudes. To permit the entropy increase associated with the amplitudes, the phases must *reduce* their entropy from the maximum value associated with uniform distribution. This reduction is associated with the formation of flow structure. We shall see that structures form suddenly as the parameter $D_{\mathbf{k}\mathbf{p}\mathbf{q}}$ is slowly decreased. Although we are always very far away from the absolute equilibrium regime represented by (2.31), this sudden appearance of structure resembles the phase changes described by equilibrium statistical mechanics.

3. Entropy in EDQNM

As it is usually presented, EDQNM, (2.18), is a closed set of equations in the averages of the spectral amplitudes $\langle A_{\mathbf{k}} \rangle$. In the loose derivation of Section 1, we have omitted the

averaging operators from the amplitudes, but now we consider them to be restored. From this new point of view, (2.6) and (2.17) are closed equations in the statistical variables $\langle A_{\mathbf{k}} \rangle$ and $\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle$. We can go further, replacing the slowly-varying approximation (2.17) by an evolution equation for $\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle$ that is more faithful to (2.13). For example,

$$\frac{d}{dt} \langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle = -\frac{B_{\mathbf{k}\mathbf{p}\mathbf{q}}}{2} (1 - \langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle^2) - D_{\mathbf{k}\mathbf{p}\mathbf{q}} \langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle. \quad (3.1)$$

The time-independent solution of (3.1) agrees with (2.17) when $D_{\mathbf{k}\mathbf{p}\mathbf{q}}$ is large. For small $D_{\mathbf{k}\mathbf{p}\mathbf{q}}$, $\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle$ shares the property of (2.15) that it approaches +1 if $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ is negative and -1 if $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ is positive. Moreover, (3.1) respects the realizability condition $|\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle| \leq 1$. For a further discussion of (3.1), see the Appendix.

If we adopt (3.1) in place of (2.17), then the closure consists of coupled evolution equations, (2.6) and (3.1), in the statistical variables $\langle A_{\mathbf{k}} \rangle$ and $\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle$, which now enter the approximation on an equal footing. These statistical variables are what Kaneda (2007) calls the “representatives” of the closure. In his opinion, the choice of representatives is the most important step in the construction of a theory. As he states, “. . . the same closure equations can be derived by several ways of reasoning, while similar ways of reasoning result in different closure equations for different representatives. These facts show that what makes the difference between closures is the choice of representatives, rather than the method of derivation.”

However, every closure theory must pass a crucial test of self-consistency. At every time, knowledge of the theory’s representatives—in our case the $\langle A_{\mathbf{k}} \rangle$ ’s and $\langle \cos \xi_{\mathbf{k},\mathbf{p},\mathbf{q}} \rangle$ ’s—corresponds to an inexact knowledge of the system’s precise state. The volume of phase space that is consistent with the values of the representatives measures our ignorance of the precise system state. Entropy is the logarithm of that volume. By the principle of mixing in phase space, the entropy must, in the absence of external forcing and damping, increase with time. For any given closure theory, the entropy can, in principle, be expressed as a function of the theory’s representatives. Then the closure equations can be tested to see if they obey the principle of entropy increase. Information theory provides the means of calculating the entropy associated with given values of the representatives.

To illustrate the essential idea, suppose that the system consists of one amplitude A and one phase ϕ . Let $P(A, \phi)$ be the joint probability distribution of A and ϕ . It must satisfy

$$\iint A \, dA \, d\phi \, P(A, \phi) = 1. \quad (3.2)$$

Let $\langle f(A, \phi) \rangle$ be the single representative, where f is an arbitrarily chosen function. According to information theory, the probability distribution associated with the given value $\langle f \rangle$ is that distribution which maximizes

$$S = - \iint A \, dA \, d\phi \, P \ln P \quad (3.3)$$

subject to the constraint

$$\iint A \, dA \, d\phi \, f P = \langle f \rangle \quad (3.4)$$

and the normalization requirement (3.2). This variational problem leads to

$$P(A, \phi) = C \exp(-\alpha f(A, \phi)), \quad (3.5)$$

where C is the normalization constant and the Lagrange multiplier α is chosen to satisfy (3.4). If $\alpha(\langle f \rangle)$ can be determined, then (3.5) can be substituted back into (3.3) to give an

expression for the entropy $S(\langle f \rangle)$ as a function of the representative. To be consistent with the idea that mixing in phase space always degrades information, the closure equation for the evolution of $\langle f \rangle$ must satisfy

$$\frac{d}{dt} S(\langle f \rangle) > 0. \quad (3.6)$$

In the case of EDQNM, the representatives are $\langle A_{\mathbf{k}} \rangle$ for every \mathbf{k} , and $\langle \cos \xi_{[\mathbf{k}\mathbf{p}\mathbf{q}]} \rangle$ for every triad $[\mathbf{k}\mathbf{p}\mathbf{q}]$. The analog of (3.5) is

$$P[A_{\mathbf{k}}, \phi_{\mathbf{k}}] = P_A[A_{\mathbf{k}}] P_\phi[\phi_{\mathbf{k}}], \quad (3.7)$$

where

$$P_A = \prod_{\mathbf{k}} C_{\mathbf{k}} \exp(-\alpha_{\mathbf{k}} A_{\mathbf{k}}) \quad (3.8)$$

and

$$P_\phi = C_\phi \exp\left(-\sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} \beta_{[\mathbf{k}\mathbf{p}\mathbf{q}]} \cos(\phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}})\right). \quad (3.9)$$

The Lagrange multipliers are $\alpha_{\mathbf{k}}$ and $\beta_{[\mathbf{k}\mathbf{p}\mathbf{q}]}$. With no loss in generality, we take each sub-distribution to be separately normalized. Because of the factorization property (3.7), the entropy

$$S = - \iint \cdots \int \prod_{\mathbf{k}} A_{\mathbf{k}} dA_{\mathbf{k}} d\phi_{\mathbf{k}} P \ln P = S_A + S_\phi \quad (3.10)$$

is the sum of the entropy

$$S_A = - \iint \cdots \int \prod_{\mathbf{k}} A_{\mathbf{k}} dA_{\mathbf{k}} P_A \ln P_A, \quad (3.11)$$

associated with the amplitudes, and the entropy,

$$S_\phi = - \iint \cdots \int \prod_{\mathbf{k}} d\phi_{\mathbf{k}} P_\phi \ln P_\phi, \quad (3.12)$$

associated with the phases. The factorization (3.8) of P_A into distributions of a single representative is a great convenience. By easy calculations we find that $\alpha_{\mathbf{k}} = 2/\langle A_{\mathbf{k}} \rangle$, $C_{\mathbf{k}} = 4/\langle A_{\mathbf{k}} \rangle^2$, and

$$S_A = \sum_{\mathbf{k}} 2 \ln \langle A_{\mathbf{k}} \rangle \quad (3.13)$$

to within irrelevant additive constants.

The determination of S_ϕ as a function of the representatives is much more difficult: the equations that determine the $\beta_{[\mathbf{k}\mathbf{p}\mathbf{q}]}$'s as functions of the $\langle \cos(\phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}}) \rangle$'s are highly coupled. In the Appendix we offer an approximate method for calculating S_ϕ based on (3.1). However, the evolution equation for

$$S_\phi = - \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} \int d\xi_{[\mathbf{k}\mathbf{p}\mathbf{q}]} P(\xi_{[\mathbf{k}\mathbf{p}\mathbf{q}]}) \ln P(\xi_{[\mathbf{k}\mathbf{p}\mathbf{q}]}) \quad (3.14)$$

is easily obtained from the Fokker Planck equation (2.13). Suppressing vector subscripts, let

$$S = - \int d\xi P(\xi) \ln P(\xi) \quad (3.15)$$

be the phase entropy associated with a single triad. Then (2.13) implies

$$\frac{dS}{dt} = B \langle \cos \xi \rangle + D \int d\xi \frac{1}{P} \left(\frac{\partial P}{\partial \xi} \right)^2. \quad (3.16)$$

Now we collect results. Restoring the viscosity terms, we find that the rate of change of the entropy associated with the amplitudes is

$$\frac{dS_A}{dt} = \sum_{\mathbf{k}} \frac{2}{\langle A_{\mathbf{k}} \rangle} \frac{d\langle A_{\mathbf{k}} \rangle}{dt} = - \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} B_{\mathbf{k}\mathbf{p}\mathbf{q}} \langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle - \sum_{\mathbf{k}} 2\nu k^2. \quad (3.17)$$

For the entropy change associated with the phases, we obtain from (3.14) and (3.16)

$$\frac{dS_{\phi}}{dt} = \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} B_{\mathbf{k}\mathbf{p}\mathbf{q}} \langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle + \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} D_{\mathbf{k}\mathbf{p}\mathbf{q}} \int d\xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \frac{1}{P(\xi_{\mathbf{k}\mathbf{p}\mathbf{q}})} \left(\frac{\partial P}{\partial \xi}(\xi_{\mathbf{k}\mathbf{p}\mathbf{q}}) \right)^2. \quad (3.18)$$

Thus the total entropy of the stochastic model associated with EDQNM obeys

$$\frac{dS}{dt} = \frac{dS_A}{dt} + \frac{dS_{\phi}}{dt} = \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} D_{\mathbf{k}\mathbf{p}\mathbf{q}} \int d\xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \frac{1}{P(\xi_{\mathbf{k}\mathbf{p}\mathbf{q}})} \left(\frac{\partial P}{\partial \xi}(\xi_{\mathbf{k}\mathbf{p}\mathbf{q}}) \right)^2 - \sum_{\mathbf{k}} 2\nu k^2. \quad (3.19)$$

The white noise term in (2.9) corresponds to the $D_{\mathbf{k}\mathbf{p}\mathbf{q}}$ -terms in (3.18)-(3.19) and is the source of entropy in the model. It directly increases the entropy of the phases. The phase entropy is transferred to amplitude entropy by the $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ -terms in (3.17) and (3.18). The resulting increase in S_A is associated with the spreading of the energy spectrum in wavenumber space. The viscosity term—the last term in (3.19)—is the entropy sink.

If viscosity is switched off, then S_A steadily increases as the energy and enstrophy spread to higher and lower wavenumbers in the spectrum. However, if the wavenumbers are truncated to a finite set, then the spectrum evolves to the state (2.31) of maximum S_A . This corresponds to

$$\langle A_{\mathbf{k}} \rangle^2 = \frac{2}{\alpha k^2 + \beta k^4}, \quad (3.20)$$

which is equivalent to (2.31). For amplitudes of the form (3.20), the $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ defined by (2.10) vanish. The entropy transfer from the phases to the amplitudes therefore also vanishes. Phase entropy builds up until the phases reach their maximum-entropy state of uniform distribution. In this state, $P(\xi)$ is constant, $\partial P/\partial \xi$ therefore vanishes, and the entropy source—the last term in (3.18)—turns off. This absolute equilibrium state of amplitudes given by (3.20) and independent, uniformly distributed phases is of course far outside the regime of physical interest. The physically interesting cases are those in which entropy flows continuously through the system, from the phases to the amplitudes, to be finally destroyed by viscosity. The transfer of entropy from the phases to the amplitudes requires that $\langle \cos \xi_{\mathbf{k}\mathbf{p}\mathbf{q}} \rangle$ and $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ be nonvanishing and have opposite signs.

The white noise forcing of the phases, which represents the random action of the fluid on itself, determines the rate at which entropy flows through the system. The intensity of this forcing reflects the level of the turbulence in the flow. In numerical simulations of freely decaying turbulence, coherent vortices are observed to form as the turbulence between vortices subsides. This suggests that the formation of the vortices might be associated with decreasing $D_{\mathbf{k}\mathbf{p}\mathbf{q}}$ in the EDQNM model. In the following section, we investigate this possibility.

4. Monte Carlo computations

In this section we construct flow fields $\psi(x, y)$ that are consistent with the EDQNM model in the limit $D_{\mathbf{k}\mathbf{p}\mathbf{q}} \rightarrow 0$. We take the amplitudes $A_{\mathbf{k}}$ as given. That is, we specify the energy spectrum $E(k)$ of the flow. Then, according to EDQNM, the probability distribution of the phases is

$$P[\phi_{\mathbf{k}}] = C \prod_{[\mathbf{k}\mathbf{p}\mathbf{q}]} \exp(-B_{\mathbf{k}\mathbf{p}\mathbf{q}} \cos(\phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}})/D_{\mathbf{k}\mathbf{p}\mathbf{q}}), \quad (4.1)$$

where C is the normalization constant, and $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$, defined by (2.10), is determined by the specified amplitude values. The distribution (4.1) is the product of the distributions given by (2.14), one for every triad in the system. It is a complicated distribution because each phase appears in many triads. For simplicity sake (and with the justification offered below) we take $D_{\mathbf{k}\mathbf{p}\mathbf{q}} = D_0$, a constant. Then (4.1) takes the form

$$P[\phi_{\mathbf{k}}] = C e^{-H/D_0}, \quad (4.2)$$

where

$$H = \sum_{[\mathbf{k}\mathbf{p}\mathbf{q}]} B_{\mathbf{k}\mathbf{p}\mathbf{q}} \cos(\phi_{\mathbf{k}} + \phi_{\mathbf{p}} + \phi_{\mathbf{q}}). \quad (4.3)$$

The sum is over all the triads in the system. Our strategy is to sample the probability distribution (4.2) for states $\{\phi_{\mathbf{k}}\}$ consisting of a value for every Fourier phase in the flow. Each such state, combined with the prescribed amplitudes $\{A_{\mathbf{k}}\}$, corresponds to a snapshot $\psi(x, y)$ of the flow.

The limit $D_0 \rightarrow 0$ corresponds to subsiding intensity of the turbulence. Since the energy spectrum $E(k)$ remains fixed, this limit corresponds to an increasingly long time over which the system has evolved before arriving at the state corresponding to $E(k)$ from a more concentrated initial spectrum. In other words, $D_0 \rightarrow 0$ corresponds to increasing the time in which flow structures can form.

The distribution (4.2) has the form of the Boltzmann distribution with H playing the role of energy and D_0 playing the role of temperature. However, H is not the energy. Rather, it is the negative of the rate at which the entropy associated with the energy spectrum increases due to the transfer of energy between modes. By the discussion in the previous section, it is also the rate at which entropy is transferred from the phases to the amplitudes. Compare (4.3) to (2.28) and (3.17)-(3.18).

We consider 2π -periodic flow with $n = 128$ grid points in each direction. Experiment shows that the results do not depend sensitively on the prescribed energy spectrum. In the calculations described below, we specify the spectrum to be proportional to

$$E(k) = \frac{k^2}{(1 + (k/k_0)^5)} e^{-5k/k_c}, \quad (4.4)$$

with the constant k_0 chosen to give a maximum of $E(k)$ at $k = 4$. The exponential factor, with $k_c = n/2$, provides a smooth falloff in the dissipation range. We normalize $E(k)$ by the assumption $\langle \zeta^2 \rangle = 1$, where $\zeta = \nabla^2 \psi$ is the vorticity. In this section the angle brackets denote the spatial average over the periodic domain. These requirements fix the values of the $n^2/2$ independent $A_{\mathbf{k}}$, and, hence the values of the $B_{\mathbf{k}\mathbf{p}\mathbf{q}}$ defined by (2.10). The $n^2/2$ independent phases $\phi_{\mathbf{k}}$ will be drawn from the distribution (4.2). By the standards of direct numerical simulation, $n = 128$ corresponds to extremely poor spatial resolution. Unfortunately, the inefficiency of the Monte Carlo algorithm constrains n to be relatively small.

We adopt the ‘Metropolis algorithm’ for sampling the distribution (4.2). We begin with

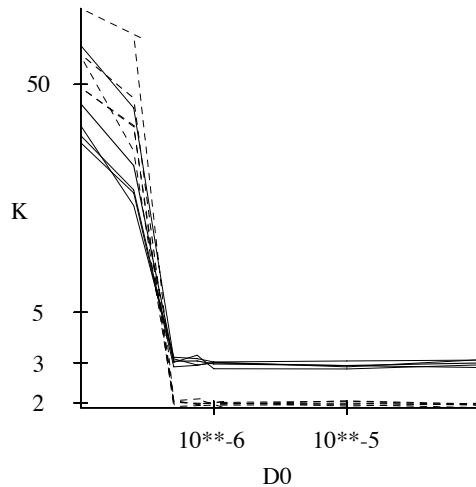


FIGURE 1. The vorticity kurtosis K_ζ (solid lines) and strain kurtosis K_σ (dashed lines) in four Monte Carlo computations in which the parameter D_0 is reduced, in stages, from $D_0 = 10^{-4}$ to 10^{-7} . The lighter lines correspond to the four separate computations, which differ only in the sequence of random numbers used in the computation. The two heavier lines represent the average of the four computations. The appearance of flow structures is associated with the rapid increase in the kurtoses that begins around $D_0 = 10^{-6}$.

an arbitrarily chosen state in which the $\phi_{\mathbf{k}}$ are independent and uniformly distributed on the interval $[-\pi, +\pi]$. The probability of this state is given by (4.2). At each step in the algorithm, we randomly perturb a single $\phi_{\mathbf{k}}$ (being careful that $\phi_{-\mathbf{k}}$ experiences the opposite perturbation), and we compute the probability of the perturbed state. If, according to (4.2) the perturbed state has a higher probability than the unperturbed state, then the perturbed state is added to our collection of states and the cycle continues. If, on the other hand, the perturbed state has a lower probability, then the ratio of probabilities is compared to a random number drawn from a distribution that is uniform on the interval $[0, 1]$. If this random number is less than the ratio of probabilities, then the perturbed state is accepted despite its lower probability. It can be shown that the set of states assembled in this manner is consistent with the probability distribution (4.2). See, for example, Kalos and Whitlock (2008).

There are $n^2/2$ values of $\phi_{\mathbf{k}}$ to be perturbed. Each phase perturbation affects $O(n^2)$ triads. The total number of triads—the number of terms in the sum (4.3)—is $O(n^4)$. At each successive value of D_0 we perform phase perturbations until the statistics of $\psi(x, y)$ cease to change. This typically requires several perturbations of every phase. Thus the number of operations required to produce a representative set of flow fields for a given D_0 is $O(n^4)$. This is what limits the size of n .

As the Metropolis algorithm proceeds, we gradually reduce the ‘temperature’ D_0 . This procedure, taken to its extreme, is called ‘simulated annealing’ and is typically used to find minima of H . In our case, the minima of H correspond to maxima in the transfer of entropy from the Fourier phases to the amplitudes. For large D_0 , the phases are nearly uniformly distributed (as in (2.14)) and there is no evidence of structures in the flow. However, as D_0 is reduced, structures form suddenly, like crystals in a supersaturated solution. Once formed, these structures prove to be surprisingly stable against further reduction in D_0 . That is, the corresponding stream function field $\psi(x, y)$ changes little

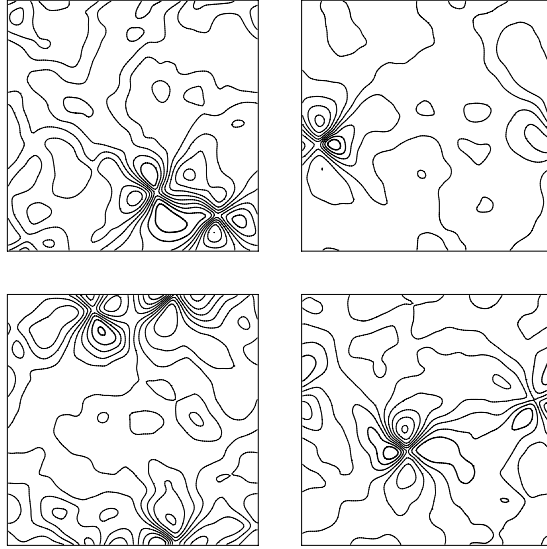


FIGURE 2. The stream function in the four independent Monte Carlo calculations at the value $D_0 = 10^{-7}$. Heavier lines correspond to larger values of the streamfunction.

as D_0 is further reduced. It is as if $H[\phi_{\mathbf{k}}]$ has many local minima—the rate of entropy increase has many local maxima—in which the system can be trapped.

The sudden appearance of structures is always associated with a sudden increase in two statistics that measure intermittency in the flow. These are the kurtoses of vorticity and strain, defined by

$$K_\zeta = \frac{\langle \zeta^4 \rangle}{\langle \zeta^2 \rangle^2}, \quad K_\sigma = \frac{\langle \sigma^4 \rangle}{\langle \sigma^2 \rangle^2}, \quad (4.5)$$

respectively. Here $\zeta^2 = (\psi_{xx} + \psi_{yy})^2$ and $\sigma^2 = (\psi_{xx} - \psi_{yy})^2 + 4\psi_{xy}^2$. The spatial averages $\langle \zeta^2 \rangle$ and $\langle \zeta^4 \rangle$ are conserved by (2.1) when $\nu = 0$, and $\langle \zeta^2 \rangle = \langle \sigma^2 \rangle$ for any periodic $\psi(x, y)$. Thus, only K_S can increase in inviscid two-dimensional flow. However, our limit $D_0 \rightarrow 0$ corresponds to a lengthening time over which viscosity can act to increase K_ζ (as is actually observed in numerical simulations of two-dimensional turbulence). We note that the viscosity appears in the exact amplitude equation (2.4), but not in the exact phase equation (2.8) or in the stochastic model (2.9)-(2.12). We also note that, in the inviscid limit, EDQNM conserves $\langle \zeta^2 \rangle$ but makes no prediction about $\langle \zeta^4 \rangle$.

Figure 1 shows the vorticity kurtosis K_ζ (solid lines) and strain kurtosis K_σ (dashed lines) in four completely independent Monte Carlo computations in which D_0 is reduced, in stages, from $D_0 = 10^{-4}$ to 10^{-7} . The lighter lines correspond to the kurtosis values in the four calculations, and the heavier lines represent the average of the four. The four calculations differ only in the sequence of random numbers used by the Monte Carlo algorithm. Each of the four calculations terminates with $\psi(x, y)$ ‘frozen’ into a state exhibiting a small number of vorticity quadrupoles. These terminal states are shown in Figures 2 and 3.

The appearance of the quadrupoles is associated with the increase in the kurtoses that begins around $D_0 = 10^{-6}$. This is far below the $D_0 = O(1)$ value stipulated by (2.20)-(2.21) for our choice of $\langle \zeta^2 \rangle = 1$. We note that (2.20)-(2.21) predict that $\mu(\mathbf{k})$ varies only logarithmically with k in the enstrophy inertial range, where $E(k) \propto k^{-3}$, and that our

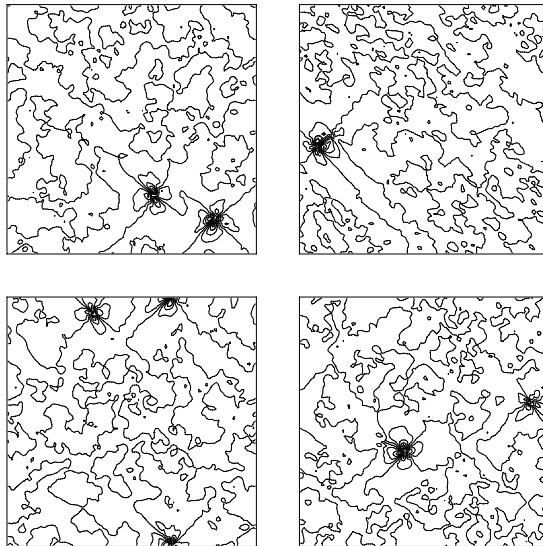


FIGURE 3. The vorticity corresponding to the stream function in Figure 2.

range of $k = 1 - 64$ is relatively small. Thus it is the size of D_0 and not its constancy that is unrealistic here. It can be shown that statistically independent, uniformly distributed phases correspond to $K_\zeta = 3$ and $K_\sigma = 2$. Figure 1 shows that these values persist right up until the rapid increase in the kurtoses that coincides with appearance of the quadrupoles.

5. Discussion

No concept generates greater controversy in turbulence theory than does the concept of entropy. This is especially true when the entropy concept is applied to turbulence that is far outside the state of ‘absolute equilibrium’ corresponding to (2.31). It is therefore worth emphasizing that all of our computational results follow from the distribution (2.14) which was obtained from the stochastic model (2.9) with no mention of entropy whatsoever. On the other hand, it would seem absurd to investigate the distribution (2.14) without offering some interpretation of the quantity $B \cos \xi$ that defines it.

The author will not hide his disappointment that the Monte Carlo computations described in Section 4 yielded only a sparse set of quadrupoles rather than the rich field of monopoles and dipoles that is actually observed in direct numerical simulations of two-dimensional turbulence, and that the quadrupoles appeared at values of D_0 that are well below the ‘normal operating range’ of EDQNM. It must reluctantly be admitted that our calculations probably have more to say about the stochastic model (2.9) than about actual two-dimensional turbulence. A better outcome evidently requires a solvable stochastic model that is more faithful to (2.8) than is (2.9).

Ayala et al (2018) find the $\psi(x, y)$ that maximizes the rate,

$$\frac{d}{dt} \iint dx dy \nabla \zeta \cdot \nabla \zeta, \quad (5.1)$$

of palinstrophy increase, for fixed values of the Reynolds number and palinstrophy, assuming periodic flow governed by (2.1). Their goal is to assess the tightness of an analytically

determined bound on (5.1). They find that the vorticity field that maximizes (5.1) consists of a single quadrupole that resembles the quadrupoles found in our calculations. The connection between their results and ours is partly explained by the following fact. In the enstrophy inertial range, $E(k) \propto k^{-3}$ corresponds to $A_{\mathbf{k}} \propto k^{-3}$. When $A_{\mathbf{k}} \propto k^{-3}$, the entropy production (2.27) of triad $[\mathbf{kpq}]$ is proportional to its palinstrophy production. Thus, for spectra of the form $E(k) \propto k^{-3}$, the rate of entropy production is proportional to the rate of palinstrophy production. For arbitrary spectra there is no direct connection between these two quantities. However, Monte Carlo calculations of the kind described in Section 4, but with various prescribed $E(k)$, all yield quadrupoles as $D_0 \rightarrow 0$. Only very occasionally does a single dipole also occur. Also, various prescribed $D_{\mathbf{kpq}}$ including the choice (2.20)-(2.21) were considered, with no significant change in results. Thus the quadrupoles seem to be a robust feature of the present approach.

One aspect of our work seems noteworthy enough to guide future work, and that is the idea that the coherent structures that occur in two-dimensional turbulence are not themselves sites of maximum entropy as suggested by several authors—see Eyink & Sreenivasan (2006, p. 97) for a concise review—but are instead low-entropy sites that exist in order to maximize the *rate* of entropy increase. The present work can be considered as a ‘toy model’ that serves to illustrate this idea. How might it be applied more realistically?

Weiss (1991) showed that (2.1) implies

$$\frac{DP}{Dt} = \begin{bmatrix} \zeta_x & \zeta_y \end{bmatrix} \begin{bmatrix} \sigma_1 & \sigma_2 + \zeta \\ \sigma_2 - \zeta & -\sigma_1 \end{bmatrix} \begin{bmatrix} \zeta_x \\ \zeta_y \end{bmatrix} \quad (5.2)$$

when $\nu = 0$. Here $P = (\zeta_x^2 + \zeta_y^2)$ is the palinstrophy density, ζ is the vorticity, and $\sigma_1 = -2\psi_{xy}$ and $\sigma_2 = \psi_{xx} - \psi_{yy}$ are the strain rates. Postulating that σ_1, σ_2 and ζ vary less rapidly than $\nabla\zeta$, Weiss concluded from (5.2) that palinstrophy increase occurs in regions of positive $W \equiv \sigma^2 - \zeta^2$, in which the squared strain rate $\sigma^2 \equiv \sigma_1^2 + \sigma_2^2$ exceeds ζ^2 . In such regions the streamfunction is hyperbolic. Regions of negative W feature vortices with closed streamlines.

In periodic flow, the spatial integral of W vanishes; on average, σ^2 and ζ^2 must cancel. It therefore seems plausible that the flow maximizes its rate of palinstrophy production (which we have associated with entropy production) by sequestering some of the vorticity in isolated vortices, so that W is positive in the broad area between vortices. The isolated coherent vortices are sites of very low entropy (very low complexity) that exist in order to maximize the rate of entropy production in the region outside the vortices. This entropy production is relatively efficient: In numerical simulations of freely decaying turbulence, the enstrophy between the vortices cascades rapidly to high wavenumber. However, the enstrophy sequestered in the isolated coherent vortices cannot be reclaimed. When the turbulence between vortices is exhausted, the entropy of the system continues to increase, but by the very slow process of infrequent vortex mergers, eventually reaching the putative maximum entropy state of two, oppositely signed, domain-filling vortices. In this scenario, the isolated coherent vortices represent ‘waste products’ of the turbulent cascade, and their agonizingly slow demise is the price to be paid for rapid entropy increase in the early stage of flow evolution.

Nothing in the foregoing interpretation anticipates the very sharp demarcation between the vortices and the turbulence that is observed to exist in high-resolution numerical simulations of two-dimensional turbulence. In these simulations, the vortices and the turbulence resemble the distinct phases that exist in a system undergoing a first-order phase transition. A principal theoretical challenge is to explain this very sharp separation. We conjecture that the concept of entropy production will find a place in this explanation.

Acknowledgments

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Appendix. Explicit expression for the phase entropy

The Fokker-Planck equation (2.13) implies

$$\frac{dx}{dt} = -B(1 - y) - Dx, \quad (\text{A.1})$$

where $x = \langle \cos \xi \rangle$ and $y = \langle \cos^2 \xi \rangle$. This is the first in an unclosed hierarchy of the equations for the moments $\langle \cos^n \xi \rangle$. Discarding the second cumulant corresponds to setting $y = x^2$ and replacing (A.1) by

$$\frac{dx}{dt} = -B(1 - x^2) - Dx. \quad (\text{A.2})$$

However, for small $|B|/D$ the steady solution of (A.2), namely $x = -B/D$, differs from the exact result (2.17) by a factor of 2. The alternative closure assumption $y = (1 + x^2)/2$ corresponds to

$$\frac{dx}{dt} = -\frac{B}{2}(1 - x^2) - Dx, \quad (\text{A.3})$$

which is equivalent to (3.1) and has the desirable properties listed thereafter.

Section 3 produced an expression (3.18) for the time derivative of S_ϕ , but we still lack an expression analogous to (3.13) for S_ϕ in terms of the representatives $\langle \cos \xi_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \rangle$. We obtain one as follows. Let $S(x)$ be the entropy associated with the knowledge of x . Then it follows from (A.3) that

$$\frac{dS}{dt} = S'(x) \left(-\frac{B}{2}(1 - x^2) - Dx \right). \quad (\text{A.4})$$

To be consistent with (3.16) we must equate

$$-S'(x) \frac{B}{2}(1 - x^2) = Bx. \quad (\text{A.5})$$

It follows that

$$S(x) = \ln(1 - x^2) \quad (\text{A.6})$$

to within an irrelevant constant. Thus the entropy of EDQNM as a function of its representatives is

$$S = S_A + S_\phi = \sum_{\mathbf{k}} 2 \ln \langle A_{\mathbf{k}} \rangle + \sum_{[\mathbf{k}, \mathbf{p}, \mathbf{q}]} \ln (1 - \langle \cos \xi_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \rangle^2). \quad (\text{A.7})$$

We see that S_ϕ takes its maximum value—zero—when the $\langle \cos \xi_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \rangle$ vanish, as when the phases are uniformly distributed.

Alternatively, if we assume the quasi-equilibrium distribution (2.14), then (3.15) implies

$$S = \ln I_0(B/D) - B/D \frac{I_1(B/D)}{I_0(B/D)}. \quad (\text{A.8})$$

To obtain $S(\langle \cos \xi \rangle)$ we must eliminate B/D between (A.8) and (2.15). In general this is difficult, but it is not hard to check that the two methods for computing S_ϕ agree when $|B|/D$ is small, and that the second method yields an expression for S_ϕ that approaches negative infinity at twice the rate of (A.7) when $|B|/D$ is large and $\langle \cos \xi \rangle$ is very close to

± 1 . We emphasize that an expression for S_ϕ is not required by our calculations, and that the equation (3.18) for its time derivative is an exact consequence of the Fokker-Planck equation (2.13).

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