Problem Set 3: Solutions

1. (Problem 2 from the notes.) Suppose particle motion evolves according to

$$X(n) = X(n-1) + \Delta V(n) \quad (1)$$
$$V(n) = (1-\alpha)V(n-1) + R(n) \quad (2)$$

where R is independent of $V(n)$, has stationary statistics and is serially uncorrelated. Find the general solution for $V(n)$. Use it to find the diffusivity of $\kappa^\infty = \lim_{t\to\infty} \frac{1}{2}\frac{d}{dt}\langle X^2(t)\rangle$ in terms of α and $\langle R^2\rangle$. Will the concentration of X-particles obey a diffusion equation? Why?

Solution: First we derive a general expression for $V(n)$:

$$\begin{aligned} V(n) &= (1-\alpha)V(n-1) + R(n) \\ &= (1-\alpha)V(n-2) + (1-\alpha)R(n-1) + R(n) \\ &= (1-\alpha)^n V(0) + \sum_{i=1}^{n} R(i)(1-\alpha)^{n-i} \end{aligned}$$

so

$$\lim_{n\to\infty} V(n) = \sum_{i=1}^{n} R(i)(1-\alpha)^{n-i}$$

Therefore

$$\begin{aligned} X(n) &= X(n-1) + \Delta V(n) \\ &= X(0) + \Delta \sum i = 1^n V(i) \\ &= X(0) + \Delta \sum_{i=1}^{n}\sum_{j=1}^{i} R(j)(1-\alpha)^{i-j} + \Delta \sum_{i=1}^{n} (1-\alpha)^i V(0) \end{aligned}$$

so

$$\lim_{n\to\infty} X(n) = X(0) + \frac{\Delta V(0)}{\alpha} + \Delta \sum_{i=1}^{n}\sum_{j=1}^{i} R(j)(1-\alpha)^{i-j}$$

where we have used the fact that $\sum_{i=1}^{\infty} x^i = 1/(1-x)$. Using this information, we can derive an expression for the diffusion. We start by defining the covariance of V:

$$\begin{aligned} C_{VV}(n,m) &= \langle V_n V_m\rangle = (1-\alpha)^{n+m} + \sum_{k=0}^{n}\sum_{l=0}^{m} R(k)R(l)(1-\alpha)^{n+m-k-l} \\ &= \sum_{k=0}^{l=\min(n,m)} \langle R(k)^2\rangle (1-\alpha)^{2l-2k}(1-\alpha)^{|n-m|} \\ &= \langle R^2(1-\alpha)^{|n-m|}\rangle \sum_{k=0}^{l=\min(n,m)} (1-\alpha)^{2k} \\ &= \langle R^2\rangle (1-\alpha)^{|n-m|}\frac{1-(1-\alpha)^{2l+2}}{1-(1-\alpha)^2} \end{aligned}$$

$$= \langle R^2\rangle(1-\alpha)^{|n-m|}\frac{1-(1-\alpha)^{2l+2}}{1-(1-\alpha)^2}$$
$$= \frac{\langle R^2\rangle(1-\alpha)^{|n-m|}}{\alpha(2-\alpha)} \text{ in the limit where } n, m \text{ are big}$$

Then we can use the covariance to compute $\langle X(n)^2\rangle$. Here we note that the constants in $X(n)$ are uncorrelated with $R(k)$. Formally $\langle X(n)^2\rangle$ computed here should also include a term of the form $(X(0) + \Delta V(0)/\alpha)^2$.

$$\begin{aligned}
\langle X(n)^2\rangle &= \Delta^2 \sum_{k=0}^{n}\sum_{l=0}^{n} V(k)V(l) = \sum_{k=0}^{n}\sum_{l=0}^{n} C_{vv}(k-l) \\
&= \Delta^2 \sum_{k=-n}^{n}(n-|k|)C_{vv}(k) \\
&= \Delta^2 \frac{\langle R^2\rangle}{\alpha(2-\alpha)}\left[n + 2\sum_{k=1}^{n}(n-k)(1-\alpha)^k\right] \\
&= \Delta^2 \frac{\langle R^2\rangle}{\alpha(2-\alpha)}\left[n + 2n(1-\alpha)\sum k = 0^{n-1}(1-\alpha)^k - 2(1-\alpha)\sum_{k=0}^{n-1}(k+1)(1-\alpha)^k\right] \\
&= \Delta^2 \frac{\langle R^2\rangle}{\alpha(2-\alpha)}\left[n + 2n(1-\alpha)\frac{1-(1-\alpha)^n}{1-(1-\alpha)} - 2(1-\alpha)\frac{1-(n-1)(1-\alpha)^n}{(1-(1-\alpha))^2}\right] \\
&= \Delta^2 \frac{\langle R^2\rangle}{\alpha^2(2-\alpha)}\left[n(2-\alpha) - 2n(1-\alpha)^{n+1} - \frac{2(1-\alpha)}{\alpha}(1-(n-1)(1-\alpha)^n)\right]
\end{aligned}$$

where we've noted that $\sum kx^k = xd/dx(1/(1-x))$. As n becomes big, all terms to the power n go to zero. For large n, the time derivative of $\langle X(n)^2\rangle$ is stable:

$$\kappa^\infty = \frac{1}{2}\frac{d\langle X(n)^2\rangle}{dt} = \frac{1}{2\Delta}\frac{d\langle X(n)^2\rangle}{dn} = \frac{\Delta\langle R^2\rangle}{2\alpha^2}$$

The same result can be derived by computing $\langle X(n)^2\rangle$ directly from the expression for $X(n)$ using a quadruple sum. You may find it interesting to confirm these results numerically. The diffusion rate κ^∞ is a constant, so we expect that the concentration of particles will obey a diffusion equation.

2. The wind fields that we used in the midterm show evidence for a clear annual cycle. Use a least squares fitting procedure to estimate the mean wind, a linear trend, and the size of the annual cycle. Explain your method and show your results for both the zonal and the meridional component of the wind.

You can refine your estimate by assuming that the uncertainty in the wind fields depends on the number of measurements averaged to produce each wind estimate (column 6 of the data). Explain how you would do this?

Solution:

To solve a system of equations $\mathbf{Gm} = \mathbf{d}$, first define the data vector $\mathbf{d}$ to be your wind measurements and define a time vector $\mathbf{t}$ that is the length of $\mathbf{d}$. Now remove all missing data from $\mathbf{d}$ and $\mathbf{t}$. Finally define $\mathbf{G}$ as

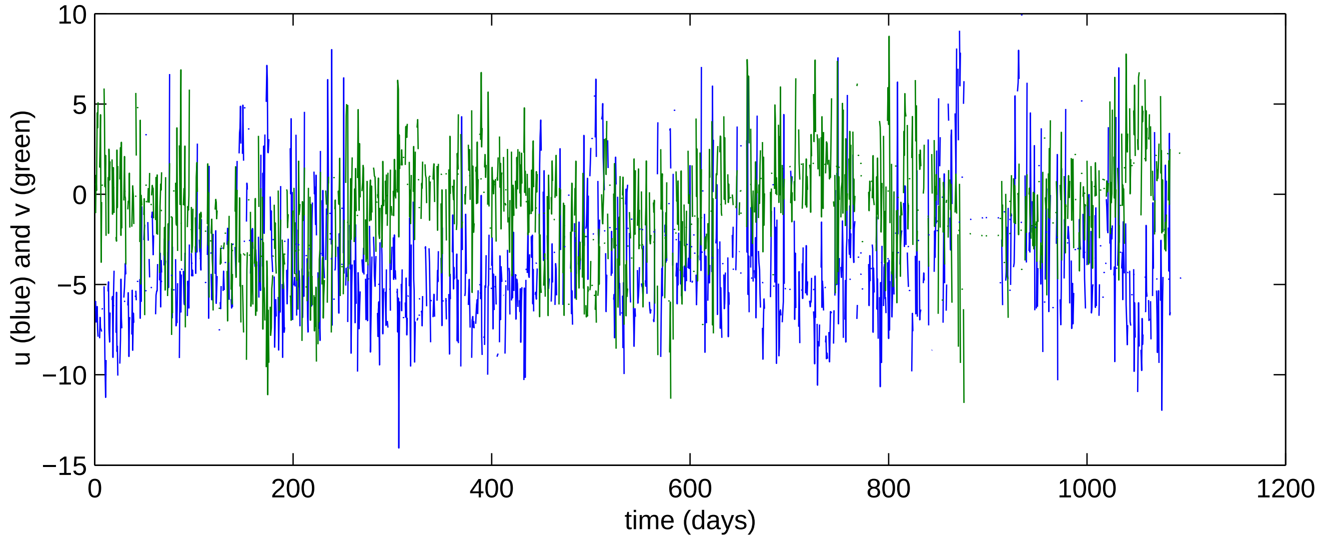
$$\mathbf{G} = \begin{bmatrix} 1 & t_1 & cos(2\pi t_1/365.25) & \sin(2\pi t_1/365.25) \\ 1 & t_2 & cos(2\pi t_2/365.25) & \sin(2\pi t_2/365.25) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_N & cos(2\pi t_N/365.25) & \sin(2\pi t_N/365.25) \end{bmatrix}$$

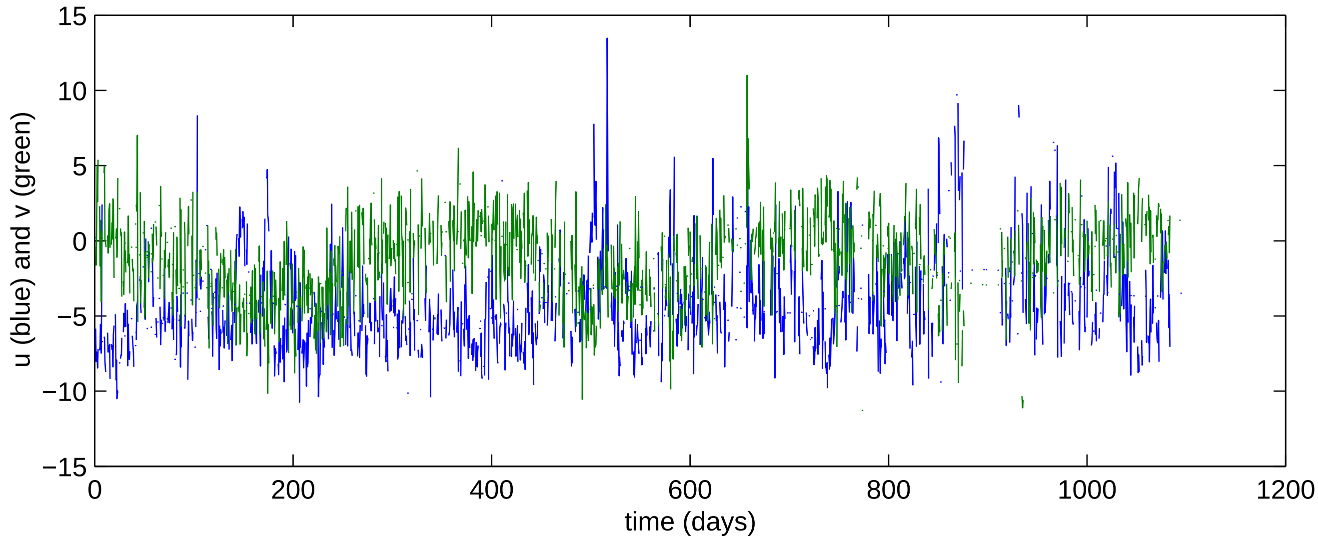
The solution vector is $\mathbf{m} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}$. Since u and v have missing data in the same places, $\mathbf{d}$ can be either u or v. The strength of the annual cycle is simply $m_3^2 + m_4^2$. Here are my solutions (from south to north): Here time is measured in days so m_2 has units of

	u_1	v_1	u_2	v_2	u_3	v_3
m_1	-4.6322	-1.5676	-5.7635	-1.8222	-6.6346	-1.2637
m_2	0.0017	0.0016	0.0030	0.0010	0.0033	0.0002
m_3	-1.8038	2.1221	-1.0365	2.0419	-0.1184	1.6228
m_4	0.4461	-0.3187	0.4953	-0.2795	0.4176	0.0596

m s^{-1} day^{-1} and the other coefficients have units of m s^{-1}.

The simplest way to refine the estimate to take account of the error estimates is to use a row weighting technique. As we saw in the midterm, the uncertainty in a mean is equal to the standard deviation, σ, divided by the square root of N, the number of data points. Although we don't know σ, we do know N. Therefore, we can define a weight vector, $\mathbf{w} = 1/\sqrt{N}$, where N is the number of measurements in column 6. Now divide each row of $\mathbf{G}$ and each value of $\mathbf{d}$ through by the corresponding value of $\mathbf{w}$. Thus, $\tilde{G}_{i,j} = G_{i,j}/w_i$ and $\tilde{d}_i = d_i/w_i$. Now solve the weighted system: $\mathbf{m} = (\tilde{\mathbf{G}}^T\tilde{\mathbf{G}})^{-1}\tilde{\mathbf{G}}^T\tilde{\mathbf{d}}$. In this case the weighted solution is modified only slightly relative to the original solution


u (blue) and v (green)
time (days)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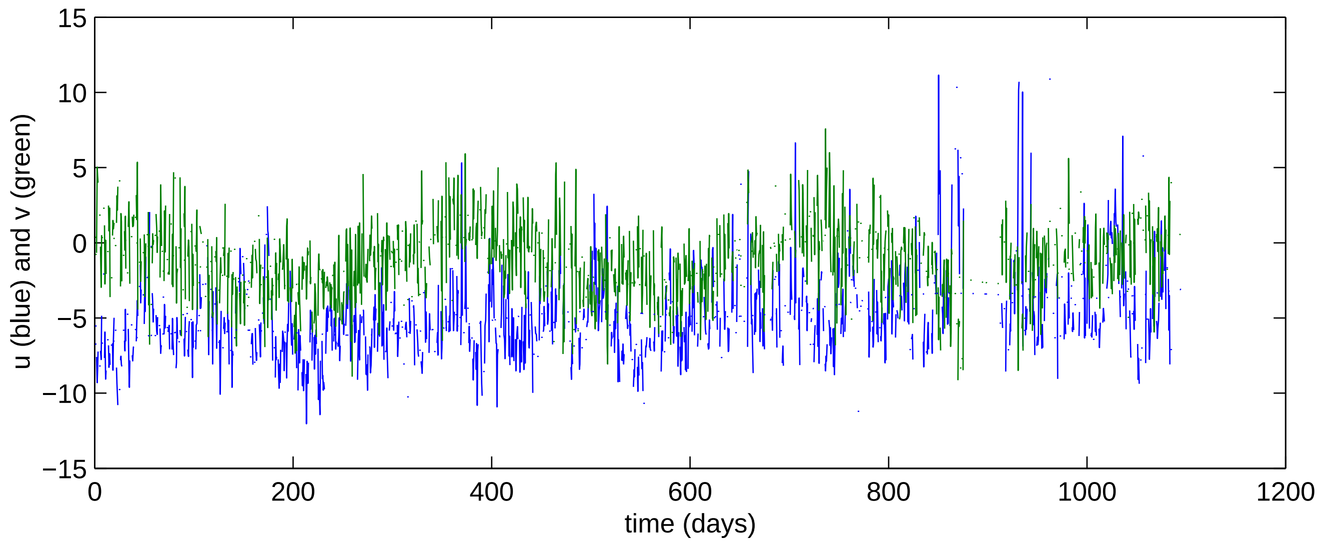
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