Coherence

When we want to know whether two time series are statistically linked, we compute correlation coefficients. Coherence provides analogous information for the Fourier transforms, telling us whether two series are statistically linked at any specific frequency. This can be important if we think that the records are noisy or otherwise uncorrelated at some frequencies, but that they also contain statistically correlated signals.

To determine the frequency-space relationship between two data sets x_n and y_n, we first divide them into segments and Fourier transform them, so that we have a set of X_k's and a set of Y_k's. When we computed spectra, we found the amplitude of each X_k and then summed over all our segments. Now we're going to do something slightly different. For each segment pair, we'll compute the product of X times the complex conjugate of Y: $X_k Y_k^*$. Then we'll sum over all the segments. In Matlab this becomes

```
sum(X.*conj(Y),2);
```

This will turn out to be a complex number. The real part is called the "co-spectrum":

$$c(\omega_k) = \Re \sum_{n=1}^{N} (X_k Y_k^*) \tag{1}$$

and the imaginary part is called the "quadrature spectrum"

$$q(\omega_k) = \Im \sum_{n=1}^{N} (X_k Y_k^*). \tag{2}$$

The corresponding amplitude is $\sqrt{c^2(\omega_k) + q^2(\omega_k)}$. For comparison the spectra for X was:

$$f_x(\omega_k) = \sum_{n=1}^{N} X_k X_k^*, \tag{3}$$

and it was always real.

The coherence resembles a correlation coefficient. It's the amplitude squared divided by the power spectral amplitudes for each of the two components:

$$C^2(\omega_k) = \frac{c^2(\omega_k) + q^2(\omega_k)}{f_x(\omega_k) f_y(\omega_k)} \tag{4}$$

It's really important that your spectra are based on more than one segment, that is that N exceeds 1. If that weren't the case, you'd just have a single realization of each spectra, and the resulting squared coherence would be

$$C^2(\omega_k) = \frac{X(\omega_k)Y^*(\omega_k)X^*(\omega_k)Y(\omega_k)}{X(\omega_k)X^*(\omega_k)Y(\omega_k)Y^*(\omega_k)} = 1, \tag{5}$$

which is not a terribly informative result. When it's done properly, coherence measures how well different segments of x and y show the same type of relationship at a given frequency.

The phase $\phi(\omega_k) = \tan^{-1}(-q(\omega_k)/c(\omega_k))$ tells us the timing difference between the two time series. If $\phi = 0$, changes in x and y happen at the same time. If $\phi = \pi$, then x is at a peak when y is at a trough. And a value of $\phi = \pi/2$ or $\phi = -\pi/2$ tells us that the records are a quarter cycle different.

How much confidence do we have in our results? For the coherence, we require that the squared coherence exceed:

$$\beta = 1 - \alpha^{1/(n_d - 1)} \tag{6}$$

where α is a measure of the significance level. If $\alpha = 0.05$ that means that there is less than a 5% chance that random noise could have produced a coherence as high as the observed value. The number of data segments used is n_d.

The phase error can seem a little murky. Formally, the uncertainty in the phase is

$$\delta_\phi = \sin^{-1}\left[t_{\alpha,2n_d}\sqrt{\frac{1-C_{xy}^2}{2n_d C_{xy}^2}}\right] \tag{7}$$

where $t_{\alpha,2n_d}$ is the "Student t distribution".

```
N=length(data);
M=segment_length/2; % define this value
for n=1:floor(N/M-1)
 d=data((n-1)*M+1:(n+1)*M); %select data for the nth segment
 d2=data2((n-1)*M+1:(n+1)*M); %select data for the nth segment
 fd(:,n)=fft(d);        % compute fft
 fd2(:,n)=fft(d2);        % compute fft
end
sd=sum(abs(fd(1:M+1,:)).^2,2)/N; % sum over all spectra (sum over 2nd index)
sd(2:end)=sd(2:end)*2;
sd2=sum(abs(fd2(1:M+1,:)).^2,2)/N; % sum over all spectra (sum over 2nd index)
sd2(2:end)=sd2(2:end)*2;
cd=sum(fd(1:M+1,:).*conj(fd2(1:M+1,:)),2)/N;
cd(2:end)=cd(2:end)*2;  % since we multiplied the spectra by 2, we also
                        % need to multiply the cospectrum by 2

nd=floor(N/M-1);
C=abs(cd)./sqrt(sd.*sd2);
delta_C = sqrt(1-alpha^(1/(nd-1)));

phase = atan2(-imag(cd),real(cd));
delta_phase = asin(tinv(0.975,2*nd)*...
         sqrt((1-C.^2)./(2*nd*C.^2)));
```

Example: Coherence and Wave Spectra

So let's see whether surfboard acceleration measurements show any signs of coherence. We'll start by comparing vertical and horizontal accelerations of the free floating accelerometer, as shown in Figure 1. These two records have rather different spectra as shown in Figure 2. The two records are coherent, as shown in Figure 3 with a phase difference of roughly π radians, implying that they are 180° out of phase, at least at the frequencies at which they are actually coherent. In contrast, the vertical acceleration for the free floating accelerometer is not coherent with vertical acceleration from the shortboard.

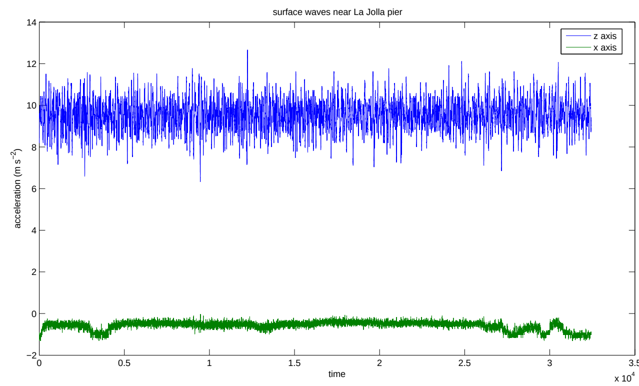

Figure 1: Time series of vertical acceleration and x-axis accleration for free-floating accelerometer near Scripps pier.

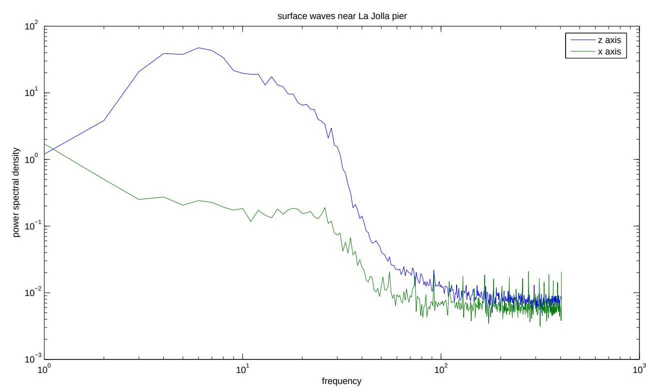

Figure 2: Spectra for vertical and x acceleration of free-floating accelerometer near Scripps pier.

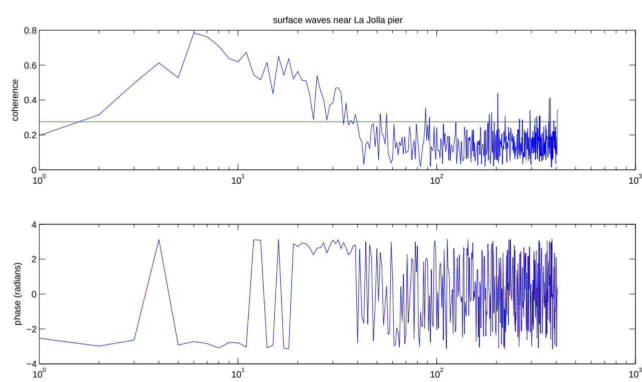

Figure 3: (top) Coherence of vertical and x acceleration of free-floating accelerometer near Scripps pier. (bottom) Phase difference between vertical and x acceleration components.

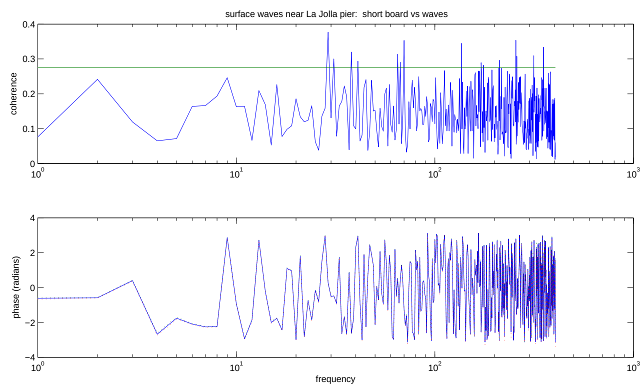

Figure 4: (top) Coherence of vertical acceleration of free-floating accelerometer versus shortboard accelerometer near Scripps pier. (bottom) Phase difference.