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Forced-dissipative two-dimensional turbulence: A scaling regime controlled by drag

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We consider two-dimensional turbulence driven by a steady prescribed sinusoidal body force working at an average rate ε. Energy dissipation is due mainly to drag, which damps all wave number at a rate μ. Simulations at statistical equilibrium reveal a scaling regime in which $\varepsilon \propto \mu^{1/3}$, with no significant dependence of ε on hyperviscosity, domain size, or numerical resolution. This power-law scaling is explained by a crude closure argument that identifies advection by the energetic large-scale eddies as the crucial process that limits ε by disrupting the phase relation between the body force and fluid velocity. The average input ε is due mainly to spatial regions in which the large-scale velocity is much less than the root-mean-square velocity. We argue that $\varepsilon \propto \mu^{1/3}$ characterizes energy injection by a steady or slowly changing spectrally confined body force.

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We consider two-dimensional turbulence [1] driven by a steady body force at small scales and study the dependence of the energy injection rate ε on external control parameters. The power ε is the most important statistical quantity characterizing forced-dissipative two-dimensional turbulence and plays a fundamental role in Kraichnan's theory of the inverse energy cascade [2]. While it is common to fix ε in theoretical and numerical studies [3–6], ε is not a prescribed parameter in many situations. For example, in laboratory experiments with electromagnetic driving [7–9], ε is the product of an unknown fluid velocity and the Lorentz force. More broadly, the statistics of energy injection in dissipative nonequilibrium systems is a frontier issue in statistical mechanics [10,11]. Here we are concerned with the most basic statistic, namely, the mean power delivered to the fluid by an imposed force.

To achieve statistically steady two-dimensional turbulence, with an inverse cascade, energy must be removed at length scales much larger than those of the forcing. Thus drag is essential because it is the only natural dissipative mechanism acting on large-scales [5,9,12–17]. We investigate the dependence of ε on the coefficient of drag μ and report a small-drag regime with $\varepsilon \propto \mu^{1/3}$. This implies an important degree of spectral nonlocality [3], else small drag, acting mainly at small wave numbers, would not also affect energy injection at much higher wave numbers. Thus the dependence of ε on drag is apparently contrary to Kraichnan's phenomenology [2] that energy transfers in the $k^{-5/3}$ inverse cascade are local. Reconciling this finding with Ref. [2] is a secondary goal of this Rapid Communication.

As a model of two-dimensional hydrodynamics, consider

$$\zeta_t + u\zeta_x + v\zeta_y = \tau_f^{-2}\cos(k_f x) - \mu\zeta - \nu\nabla^8\zeta. \tag{1}$$

In Eq. (1) the incompressible velocity field is obtained from a stream function $\psi(x,y,t)$ according to $(u,v)=(-\psi_y,\psi_x)$ and the vorticity is $\zeta \equiv \psi_{xx}+\psi_{yy}$. The vorticity forcing is the curl of a body force $f(x) \equiv \tau_f^{-2}k_f^{-1}\sin(k_f x)$ in the y component of the momentum equation. The domain is a doubly periodic square $2\pi L \times 2\pi L$, where $k_f L$ is an integer. Figure 1 shows a typical solution.

The dissipation in Eq. (1) is drag μ, and scale selective "hyperviscosity" ν that removes enstrophy at high wave numbers. Sinusoidal forcing on the right of Eq. (1) is the suggestion of Kolmogorov as the simplest representative of narrow-band forcing (i.e., forcing confined to an annulus in wave-number space) and is the protocol used in some experiments [9,13]. Other experiments [7,8] employ electromagnetic driving so that ε is the time and space average of an imposed narrow-band force f times v [see Eq. (3) below].

Equation (1) has a steady laminar solution,

$$\zeta_{\mathsf{L}}(x) = \frac{\cos k_f x}{\tau_f^2(\mu + \nu k_f^8)}. \tag{2}$$

If $\nu k_f^8 \tau_f \ll 1$ then the stability of this laminar solution is controlled by drag: if $0.52 \le \mu\tau_f$ then ζ_{L} is linearly stable [19,20]. In pursuit of two-dimensional turbulence we have

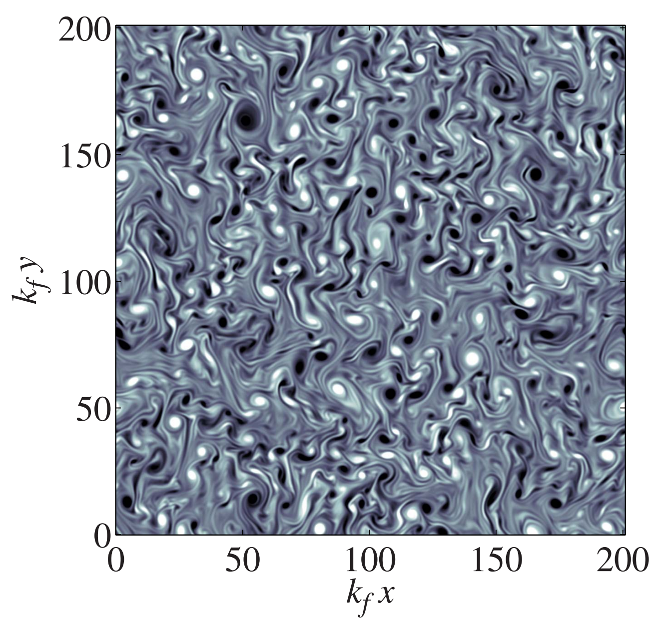

FIG. 1. (Color online) A snapshot of the vorticity using a nonuniform contour level to show the low-level filaments between the vortices. This is a solution of Eq. (1) with $\mu\tau_f=0.004$, $k_f^8\tau_f\nu=10^{-5}$, and $k_f L=32$. The resolution is 1024^2, and the time-stepping uses the fourth-order Runge-Kutta scheme in [18].

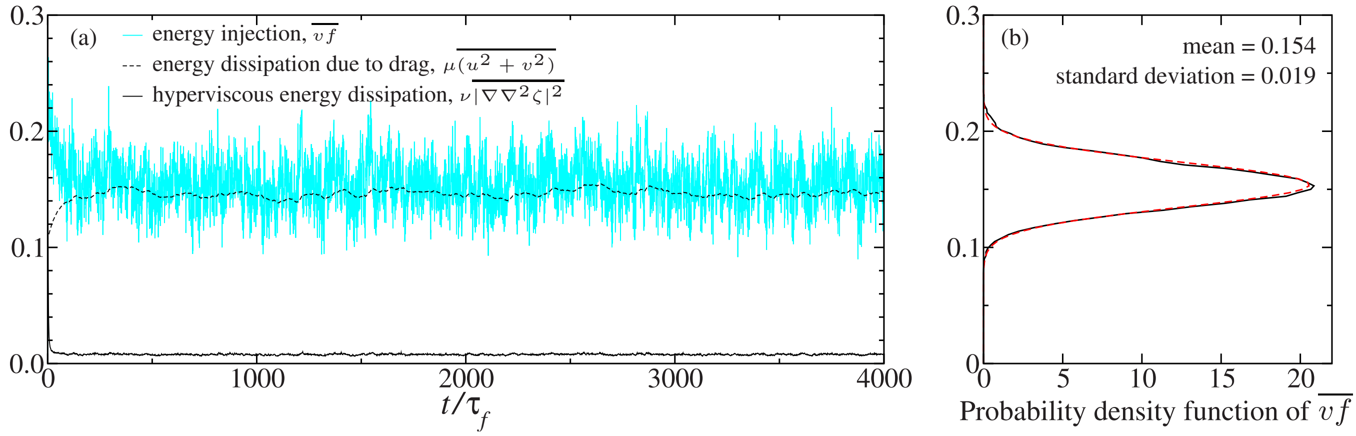

FIG. 2. (Color online) (a) Time series of three instantaneous quantities for the run in Fig. 1. The overbar denotes a spatial average over the domain. (b) The solid curve is the probability density function of $\overline{vf}$ using the data after $t_0=500$. The dashed curve is a Gaussian.

obtained statistically steady solutions of Eq. (1) in the supercritical regime $0.001 \le \mu\tau_f \le 0.1$.

The vorticity in Fig. 1 consists of widely separated, almost axisymmetric vortices. These coherent structures coexist with a sea of filamentary vorticity. Within the sea, the sinusoidal signature of the steady forcing in Eq. (1) is evident as streaks roughly elongated along the y axis. Like signed vortices tend to aggregate into clusters with large-scale closed streamlines delineating regions of cyclonic and anticyclonic circulation. Although the forcing in Eq. (1) is anisotropic, the large-scale eddies, which contain most of the kinetic energy, are isotropic; space-time averages $\langle u^2\rangle$ and $\langle v^2\rangle$ typically differ by less than 1%.

If $\mu\tau_f$ is sufficiently small then an inverse cascade develops. However if $\mu\tau_f$ is too small then the inverse cascade

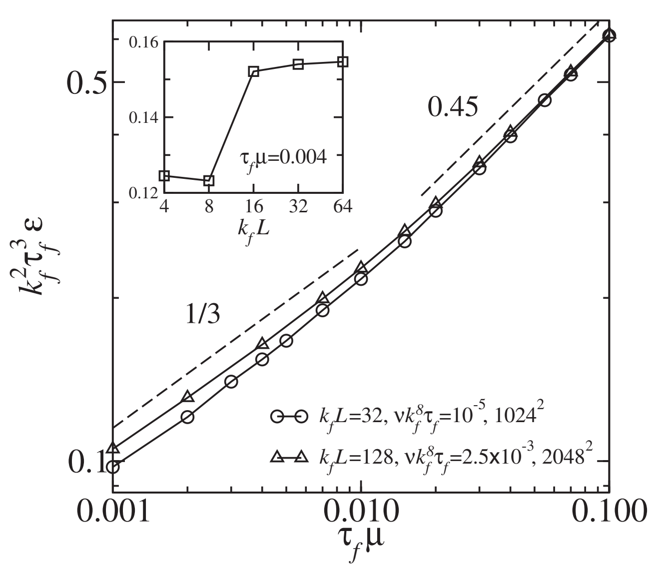

FIG. 3. Dependence of ε on μ; the results are insensitive to large changes in ν and L. A least-squares fit to the runs with $\mu\tau_f \le 0.01$ in Fig. 3 gives an exponent 0.35 for the sequence with $k_fL=32$ and 0.33 for the sequence with $k_fL=128$. The insert shows the dependence of ε on L, with $\mu\tau_f=0.004$ and $\nu k_f^8\tau_f=10^{-5}$. If $k_fL \ge 16$, ε is insensitive to L; the runs with $k_fL \le 8$ result in a condensate [21].

proceeds until energy accumulates at the domain scale, L and a "condensate" forms [21]. In the solutions discussed here we avoid these finite-size effects by ensuring that $\mu\tau_f$ and k_fL are large enough so that the inverse cascade is halted at a length scale significantly less than L [5,12]. With $k_fL=32$, we find that $0.001 \le \mu\tau_f \le 0.1$ is small enough to allow an inverse cascade, yet large enough to prevent condensation.

The average energy injection is

$$\varepsilon \equiv \langle vf\rangle = \frac{1}{(2\pi L)^2 t_1}\int_{t_0}^{t_0+t_1}\int\int vf\,dx\,dy\,dt, \qquad (3)$$

where $v=\psi_x$ and $f(x) \equiv \tau_f^{-2}k_f^{-1}\sin(k_f x)$ is the body force in the y component of the momentum equations. In Eq. (3), t_0 is selected so that the system is in statistical steady state, and t_1 must be long compared to large-eddy turnover time, which scales as μ^{-1} [12]. Figure 2 shows that this time average is necessary to remove turbulent fluctuations in the instantaneous injection. Using the space-time average in Eq. (3), the steady-state energy power integral obtained from Eq. (1) is

$$\varepsilon = \mu\langle u^2+v^2\rangle + \nu\langle|\boldsymbol{\nabla}\nabla^2\zeta|^2\rangle. \qquad (4)$$

As illustrated in Fig. 2, most of the energy dissipation is due to drag so that there is a dominant balance between the first and second terms in Eq. (4).

Figure 3 summarizes the dependence of ε on μ. The results indicate power-law dependence of $k_f^2\tau_f^3\varepsilon$ on $\mu\tau_f$, with a change in the exponent when $\mu\tau_f$ is between 0.01 and 0.015. The dependence of ε on μ is insensitive to changes in L, ν, and numerical resolution, provided finite-size effects are avoided.

The low-drag regime, $\mu\tau_f \le 0.01$ in Fig. 3, is characterized by $\varepsilon \sim \mu^{1/3}$ and by a substantial transfer of energy to $k \ll k_f$, e.g., by a spectral peak at $k \le 0.1k_f$ in Fig. 4. Thus, based on the numerical solutions, we hypothesize that the energy injection is

$$\varepsilon = 0.97k_f^{-2}\tau_f^{-8/3}\mu^{1/3}, \quad \text{as} \quad \mu\tau_f \to 0, \qquad (5)$$

0.97 fits the sequence with $k_fL=32$ in Fig. 3.

Marcus *et al.* [22] presented scaling arguments indicating that $\varepsilon \sim k_f^{-2}\tau_f^{-3}$ as $\mu\tau_f \to 0$: in contrast to Eq. (5), ε is pre-

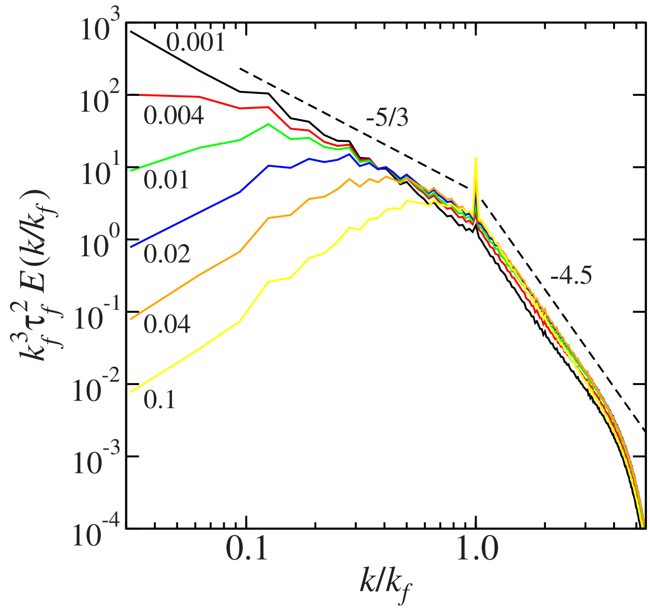

FIG. 4. (Color online) Energy spectra at different $\mu\tau_f$ for selected runs with $k_f L=32$. The change in exponent at about $\mu\tau_f=0.015$ in Fig. 3 coincides with the formation of a $k^{-5/3}$ range.

dicted to be asymptotically independent of drag. Indeed that is the expectation based on the usual phenomenological arguments [2,12]. Of course, one cannot rule out the possibility that our computations have not reached this asymptopia. However the computations summarized in Fig. 3 convince us that Eq. (5) robustly characterizes a significant portion of the low-drag large-domain parameter space.

In the energy power integral Eq. (4), ε is mostly balanced by drag dissipation, with only small hyperviscous dissipation (less than 10% in the runs with $k_f L=32$). Thus Eqs. (4) and (5) imply that $\sqrt{\langle u^2+v^2\rangle}\sim\mu^{-1/3}$.

Kolmogorov's dimensional arguments [2] predict that the energy spectrum of the inverse cascade is

$$E(k) = C\varepsilon^{2/3}k^{-5/3}. \tag{6}$$

Substituting Eq. (5) into Eq. (6) shows that the spectral level varies as $\mu^{2/9}$. Although the dependence of the spectral level on k_f, τ_f, and μ undermines Kolmogorov's dimensional argument, in the low-drag regime the observed energy spectra does exhibit a $k^{-5/3}$ regime (see Fig. 4). Thus it seems that the dependence of ε on a nontrivial combination of the external parameters in Eq. (5) does not violate the dimensional arguments leading to Eq. (6).

To summarize, our computations suggest the existence of a scaling regime of body-forced two-dimensional turbulence in which important large-scale properties of the flow follow from Eqs. (5) and (6). In the laminar solution Eq. (2), the energy injection is proportional to μ^{-1}, which is much greater than the $\mu^{1/3}$ turbulent injection in Eq. (5). The scaling argument given in Ref. [22] results in $\varepsilon\sim\mu^0\gg\mu^{1/3}$. Thus the simulated turbulent flow is relatively inefficient at extracting energy from $f(x)$. This inefficiency is because turbulent energy extraction is limited by the following two processes:

(1) Nonlinear transfer of energy, mostly to modes with wave number $k\ll k_f$.

(2) Random advection by the energetic modes reduces the spatial correlation between $f(x)$ and the k_f components of the flow.

Process 1 drives the inverse cascade: modes with $k\ll k_f$ are parasitic on the forced modes. Via process 2, the parasitic modes disrupt the spatial correlation between v and f, necessary in $\varepsilon=\langle vf\rangle$, to the point where the extraction of energy balances the average dissipation of the parasitic modes.

With the above insights, we formulate a random sweeping model as follows. We decompose the flow into $\zeta=\hat{\zeta}(x,t)+\tilde{\zeta}(x,y,t)$, where $\hat{\zeta}$ is the forced component of the flow and $\tilde{\zeta}$ is the remainder of the solution. The velocity of the forced mode is $\hat{v}$, so $\hat{\zeta}=\hat{v}_x$. A naive closure for the dynamics of the forced mode is

$$\hat{\zeta}_t + U\hat{\zeta}_x + V\hat{\zeta}_y = \tau_f^{-2}\cos k_f x - \eta\hat{\zeta}. \tag{7}$$

The loss of energy via process 1 is modeled by the damping term $-\eta(x,y,t)\hat{\zeta}$ in Eq. (7), i.e., the forced mode experiences additional drag, with a rate constant η. Since $\eta\gg\mu$, the drag dissipation of the forced mode is neglected. Process 2 is modeled by a slow-varying velocity, $[U(x,y,t),V(x,y,t)]$, representing the energetic large-scale eddies, that sweeps $\hat{\zeta}$ across the forcing pattern. Because the large-scale flow is isotropic $\langle U^2\rangle=\langle V^2\rangle$.

The large-scale modes and the forced modes are coupled by demanding that the energy power integral Eq. (4) is satisfied. To implement this coupling, we write $\langle u^2+v^2\rangle=\langle U^2+(V+\hat{v})^2\rangle\approx2\langle U^2\rangle$ consistent with $\eta\gg\mu$ and isotropy. Further neglecting the small hyperviscous dissipation, the energy power integral used in our model is

$$2\mu\langle U^2\rangle = \langle\hat{v}f\rangle \equiv \varepsilon. \tag{8}$$

We now compute ε for the random sweeping model. The large-eddy turnover time is of order μ^{-1}, which is much larger than the sweeping time, i.e., the time it takes for large-scale eddies with typical velocity $\sqrt{\langle U^2\rangle}$ to sweep through a distance k_f^{-1}. So solving the steady-state version of Eq. (7):

$$\hat{v} \approx \frac{\eta\sin k_f x - Uk_f\cos k_f x}{\tau_f^2 k_f[\eta^2+(Uk_f)^2]}. \tag{9}$$

The solution above is approximate because we have used the assumption that U varies on scales much larger than k_f^{-1}.

We relate the average nonlinear transfer rate $\langle\eta\rangle$ to ε by arguing that energy extraction from the forced mode is proportional to the total shear acting on wave numbers of order k_f so that the phenomenology of two-dimensional turbulence [2] implies $\langle\eta\rangle\sim(\varepsilon k_f^2)^{1/3}$, which can be compared to the sweeping time through a distance k_f^{-1}. Using Eq. (8), this argument identifies a nondimensional parameter;

$$\alpha \equiv \frac{\text{interaction rate}}{\text{sweeping rate}} = \frac{\langle\eta\rangle}{k_f\sqrt{\langle U^2\rangle}} \propto \mu^{1/2}\varepsilon^{-1/6}. \tag{10}$$

We anticipate from Eq. (5) that $\alpha\sim\mu^{4/9}\ll1$ as $\mu\tau_f\to0$.

To compute $\varepsilon \equiv \langle \hat{v} f \rangle$, we replace the space-time average by an ensemble average using a prescribed probability density function for the random variables U and η. We assume that this probability density function has the scaling form

$$P(U,\eta) = \mathcal{P}(U/\sqrt{\langle U^2\rangle}, \eta/\langle \eta \rangle)/(\sqrt{\langle U^2\rangle}\langle \eta\rangle). \tag{11}$$

In principle, $\mathcal{P}$ might be inferred by statistical analysis of the simulations, e.g., the statistics of U are Gaussian (not shown). However, the basic predictions of the closure are insensitive to the form of $\mathcal{P}(U', \eta')$.

Using Eqs. (9) and (11), $\langle \hat{v} f\rangle$ is

$$\varepsilon = \frac{1}{2\tau_f^4 k_f^3 \sqrt{\langle U^2\rangle}} \int\int \frac{\alpha \eta'}{U'^2 + (\alpha\eta')^2} \mathcal{P}(U',\eta') dU' d\eta' . \tag{12}$$

The U' integral is easily evaluated because $\alpha \ll 1$ implies that the function multiplying $\mathcal{P}$ in Eq. (12) has a much narrower peak than the width of $\mathcal{P}$, i.e., to approximately evaluate the average over U', one can replace $\mathcal{P}(U',\eta')$ by $\mathcal{P}(0,\eta')$ in Eq. (12) and integrate over U' so that

$$\varepsilon = \frac{\pi}{2\tau_f^4 k_f^3 \sqrt{\langle U^2\rangle}} \int \mathcal{P}(0,\eta') d\eta' . \tag{13}$$

Finally, eliminating $\langle U^2\rangle$ between Eqs. (8) and (13), one finds ε in Eq. (5). The numerical constant is related to variables in the closure by $0.97 = 2^{-1/3}[\pi \int \mathcal{P}(0,\eta') d\eta']^{2/3}$.

This closure shows that energy input is due largely to regions in which U is anomalously small: the integral on the right of Eq. (12) is determined mainly by the large contribution of regions in which U happens to be much smaller than $\sqrt{\langle U^2\rangle}$. Thus the spatial fluctuations in the energy injection rate play a crucial role in determining the net energy input. Our model also identifies a nonlocal interaction between the large-scale energetic eddies and the dynamics at the forcing scale: large eddies sweep small eddies past the stationary forcing pattern and so reduce the correlation essential for energy input. This process involves no energy transfer between the large and the small scales as $\langle U \hat{\psi} \hat{\zeta}_x \rangle = 0$. Therefore large-scale sweeping does no harm to spectral locality assumption which is the basis of the theory in Ref. [2].

With hindsight, one can give an alternative derivation of the $\mu^{1/3}$ scaling. The closure indicates that the decorrelation time between v and f is

$$t_{\text{sweep}} = \frac{1}{k_f \sqrt{\langle u^2 + v^2\rangle}} \approx \frac{1}{k_f}\sqrt{\frac{\mu}{\varepsilon}}, \tag{14}$$

and that the energy at k_f is $v_f^2 \sim \varepsilon t_{\text{sweep}} \sim \sqrt{\mu\varepsilon}/k_f$. But the power can also be estimated as $\langle v f\rangle \sim v_f k_f^{-1} \tau_f^{-2}$. Eliminating v_f then gives $\varepsilon \sim \tau_f^{-8/3} \mu^{1/3} k_f^{-2}$.

How universal is the scaling $\varepsilon \sim \mu^{1/3}$? The Kolmogorov forcing in Eq. (1) is doubly special because it is both steady and sinusoidal. We expect that a single sinusoid is representative of forces that drive a narrow band of wave numbers in spectral space. The assumption of a steady force is perhaps restrictive because then t_{sweep} is the only decorrelation time that limits energy input at k_f. Applying a randomly changing force, with an intrinsic decorrelation time t_{force}, provides an additional limiting mechanism. But if in pursuit of an increasingly large inertial range one takes drag $\mu \to 0$ (with fixed $t_{\text{force}} \neq 0$) then for sufficiently small μ, t_{sweep} will be less than t_{force}. In other words, for sufficiently small drag, sweeping is the mechanism that limits power input by an applied force. Thus we speculate that the low-drag $\mu^{1/3}$ regime is characteristic of energy input by narrow-band forces with fixed nonzero decorrelation time.

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