On the energy of elliptical vortices

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Consider a two-dimensional axisymmetric vortex with circulation Γ. Suppose that this vortex is isovortically deformed into an elliptical vortex. We show that the reduction in energy is $\Delta E = -\Gamma^2 \ln[(q+q^{-1})/2]/(4\pi)$, where q^2 is the ratio of the major to the minor axis of any particular elliptical vorticity contour. It is notable that ΔE is independent of the details of vorticity profile of the axisymmetric vortex and, in particular, independent of its average radius. The implications of this result for the two-dimensional inverse cascade are briefly discussed. © *2010 American Institute of Physics*. [doi:10.1063/1.3474703]

The problem of perturbed vortices, which originated with Rayleigh and Kelvin, remains of interest largely because of the central role played by localized vortices in geophysical flows and in plasma physics. Here, we consider elliptical vortices, regarded as finite amplitude distortions of an initial axisymmetric vortex, and we show that there is a simple formula for the change ΔE in vortex energy induced by the distortion.

The vorticity of an elliptical vortex can be written as

$$\omega(\boldsymbol{x}) = \Omega(\sqrt{(x/q)^2 + (qy)^2}), \tag{1}$$

where $q^2 \geq 1$ is the ratio of the major to the minor semiaxes of the vorticity contours. The function $\Omega(r)$ is arbitrary but assumed to decrease fast enough as $r \to \infty$ so that the circulation,

$$\Gamma \equiv \int \omega(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 2\pi \int_0^\infty \Omega(r) r \mathrm{d}r, \tag{2}$$

is finite. Typical models take $\Omega(r)$ to be a Gaussian[1,2] or a top-hat function.

The vorticity field (1) can be thought of as resulting from the isovortical deformation of an axisymmetric vortex. For example, consider the advection of an initially axisymmetric vortex with vorticity $\Omega(|\boldsymbol{x}|)$ by the uniform straining field $(\sigma x, -\sigma y)$. For $|\Omega| \ll |\sigma|$, the vorticity $\omega(\boldsymbol{x},t)$ is advected passively[3] and satisfies the equation

$$\omega_t + \sigma x \omega_x - \sigma y \omega_y = 0, \quad \omega(\boldsymbol{x},0) = \Omega(|\boldsymbol{x}|). \tag{3}$$

The solution at time t then takes the form (1) with $q = \exp(\sigma t)$.

More generally, approximately passive advection of a vortex by an incompressible flow with velocity depending linearly on the coordinates is the basis of rapid distortion theory, which is valid provided that the strain rate is large enough or acts over a very short time. Perturbed vortices such as (1) can be realized experimentally by manipulating columns of magnetically confined electron plasmas.[4] Once the external perturbation is switched off or is reduced to a low level, Eq. (1) defines a disturbed smooth vortex, and self-induction then leads to a nonelliptical rearrangement of the vorticity. This nonlinear relaxation is quite complicated and has been the focus of several studies.[1,5–8] A main point is that the energy of the perturbed nonelliptical vortex, evolving under self-induction from the initial condition in Eq. (1), is constant and given by the formula for ΔE in Eq. (14).

Gaussian elliptical vortices provide a simple model of the smoothly varying vortices observed in simulations of two-dimensional turbulence.[1] Although a Gaussian elliptical vortex is not a solution of the nonlinear equations of motion, this profile is often a surprisingly good approximation to observed vortices.[2]

In any event, the elliptical vortex in Eq. (1) can be written as

$$\omega(\boldsymbol{x}) = \Omega(|Q\boldsymbol{x}|), \tag{4}$$

where the 2×2 matrix $Q(t)$ satisfies $\det Q(t) = 1$. Assuming that Q is nondegenerate, a rotation of the coordinate system then reduces Q to $\mathrm{diag}(q^{-1}, q)$ and $\omega(\boldsymbol{x},t)$ to Eq. (1).

The streamfunction $\psi(\boldsymbol{x})$ is related to the vorticity $\omega(\boldsymbol{x})$ by the Poisson equation, $\omega = \psi_{xx} + \psi_{yy}$, so that in an unbounded domain, the vortex streamfunction is

$$\psi(\boldsymbol{x}) = \frac{1}{2\pi} \int \ln|\boldsymbol{x} - \boldsymbol{x}'| \omega(\boldsymbol{x}') \mathrm{d}\boldsymbol{x}'. \tag{5}$$

At large distances from the vortex center, the streamfunction is

$$\psi(\boldsymbol{x}) = \frac{\Gamma}{2\pi} \ln r + O(r^{-1}). \tag{6}$$

Because of the slow decrease of the velocity as $r \to \infty$, it is necessary to define the vortex energy as $E = -\frac{1}{2}\int \psi(\boldsymbol{x})\omega(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$.[9] Using Eq. (5), this gives

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$$E = -\frac{1}{4\pi}\int\int \ln|\boldsymbol{x}-\boldsymbol{x}'|\omega(\boldsymbol{x})\omega(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'. \tag{7}$$

Substituting $\omega(\boldsymbol{x})$ in Eq. (4) into Eq. (7), one can write the difference between the energy of the elliptical vortex and the energy of the isovortical axisymmetric vortex as

$$\Delta E = -\frac{1}{4\pi}\int\int [\Omega(|Q\boldsymbol{x}|)\Omega(|Q\boldsymbol{x}'|) - \Omega(|\boldsymbol{x}|)\Omega(|\boldsymbol{x}'|)] \times \ln|\boldsymbol{x}-\boldsymbol{x}'|\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'. \tag{8}$$

Note that this energy difference can also be interpreted as the integral of the difference between the energy densities of the two vortices, where the energy densities are understood in their usual form $\frac{1}{2}|\nabla\psi|^2$, as this integral also converges.

We now derive a simple expression for the energy difference (8). First, we introduce the change of variables $Q\boldsymbol{x}\mapsto\boldsymbol{x}$ and $Q\boldsymbol{x}'\mapsto\boldsymbol{x}'$ into the first term of Eq. (8). Because $\det Q=1$, this change of variables gives

$$\Delta E = -\frac{1}{4\pi}\int\int \Omega(|\boldsymbol{x}|)\Omega(|\boldsymbol{x}'|)\ln\frac{|Q^{-1}(\boldsymbol{x}-\boldsymbol{x}')|}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'. \tag{9}$$

Next, we make a further change of variables from $\boldsymbol{x}=r(\cos\theta,\sin\theta)$ and $\boldsymbol{x}'=r'(\cos\theta',\sin\theta')$ to (r,r',α,β), where

$$\alpha \equiv \theta-\theta', \quad \text{and} \quad \beta \equiv \tan^{-1}\left(\frac{r\sin\theta - r'\sin\theta'}{r\cos\theta - r'\cos\theta'}\right) \tag{10}$$

is the angle between $\boldsymbol{x}-\boldsymbol{x}'$ and the x-axis. To derive the Jacobian of this transformation, we first compute

$$\left|\frac{\partial(r,r',\alpha,\beta)}{\partial(x,y,x',y')}\right| = \left|\frac{\partial(r,r',\alpha,\beta)}{\partial(r,r',\theta,\theta')}\right|\left|\frac{\partial(r,r',\theta,\theta')}{\partial(x,y,x',y')}\right| = \left|\frac{\partial(\alpha,\beta)}{\partial(\theta,\theta')}\right|\left|\frac{\partial(r,\theta)}{\partial(x,y)}\right|\left|\frac{\partial(r',\theta')}{\partial(x',y')}\right| = \frac{1}{rr'}\left|\frac{\partial(\alpha,\beta)}{\partial(\theta,\theta')}\right| = \frac{1}{rr'}\left(\frac{\partial\beta}{\partial\theta}+\frac{\partial\beta}{\partial\theta'}\right).$$

We then introduce $r''=|\boldsymbol{x}-\boldsymbol{x}'|=r^2+r'^2-2rr'\cos\alpha$ and obtain from Eq. (10) that $\partial\beta/\partial\theta=(r^2-rr'\cos\alpha)/r''^2$, with an analogous expression for $\partial\beta/\partial\theta'$. Taking this into account leads to the Jacobian

$$\left|\frac{\partial(x,y,x',y')}{\partial(r,r',\alpha,\beta)}\right| = rr'. \tag{11}$$

Choosing a coordinate system such that $Q=\mathrm{diag}(q^{-1},q)$ then gives

$$\ln\frac{|Q^{-1}(\boldsymbol{x}-\boldsymbol{x}')|}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{2}\ln(q^2\cos^2\beta + q^{-2}\sin^2\beta) \tag{12}$$

and reduces Eq. (9) to

$$\Delta E = -\frac{1}{4}\int \Omega(r)r\mathrm{d}r\int \Omega(r')r'\mathrm{d}r' \times \int_0^{2\pi}\ln(q^2\cos^2\beta+q^{-2}\sin^2\beta)\mathrm{d}\beta. \tag{13}$$

The first two integrals are identical and readily expressed in terms of the circulation using Eq. (2); the third can be evaluated explicitly. This gives the final result,

$$\Delta E = -\frac{\Gamma^2}{4\pi}\ln\left(\frac{q+q^{-1}}{2}\right). \tag{14}$$

In terms of the semiaxes a and b of any particular vorticity contour, we have the equivalent form

$$\Delta E = -\frac{\Gamma^2}{4\pi}\ln\left(\frac{a+b}{2\sqrt{ab}}\right), \tag{15}$$

since $q^2=a/b$.

As announced, formula (15) shows that the energy difference between isovortical elliptical and axisymmetric vortices depends on the vorticity profile $\Omega(r)$ through the circulation Γ only. The mean radius of the vortex is also irrelevant. These remarkable properties extend the well-known result that an elliptical vortex with $\Gamma=0$ has the same energy as the equivalent isovortical axisymmetric vortex.[3,10,11]

At a practical level, the formula is valuable to evaluate the energy E of an elliptical vortex with minimal effort. We have confirmed the result by independent calculations of E for two particular vortices: the Kirchhoff ellipse with top-hat vorticity profile $\Omega(r)$ and the Gaussian vortex with Gaussian $\Omega(r)$. The necessary computations are quite involved, especially in the Gaussian case. For the Kirchhoff ellipse, the calculation reveals an omission in Lamb's[12] discussion. Thus, it is worthwhile to sketch the derivation of E and ΔE in this classic example.

The Kirchhoff elliptical vortex has uniform vorticity, $\omega(\boldsymbol{x})=\varpi$, inside the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{16}$$

and zero outside. This has the form (1) of an isovortical deformation of an axisymmetric (Rankine) vortex with $\Omega(r)=\varpi$ for $r<\sqrt{ab}$ and $\Omega(r)=0$ for $r>\sqrt{ab}$. The circulation of the Kirchhoff vortex is $\Gamma=\pi ab\varpi$. The exact solution for the streamfunction, based on the elliptical coordinates (ξ,η) defined by

$$x+\mathrm{i}y=\sqrt{a^2-b^2}\cosh(\xi+\mathrm{i}\eta), \tag{17}$$

is provided by Lamb.[12] In terms of elliptical coordinates, the ellipse in Eq. (16) is

$$\xi(\boldsymbol{x}) = \ln\sqrt{\frac{a+b}{a-b}}. \tag{18}$$

Outside the elliptical vortex, the streamfunction is

$$\psi = \frac{\Gamma}{2\pi}\left[\xi + \frac{1}{2}e^{-2\xi}\cos 2\eta + \frac{1}{2}\ln\left(\frac{a^2+b^2}{4}\right)\right], \tag{19}$$

while inside the vortex,

$$\psi = \frac{\Gamma}{2\pi ab}\frac{bx^2+ay^2}{a+b} + \frac{\Gamma}{2\pi}\left[\ln\left(\frac{a+b}{2}\right) - \frac{1}{2}\right]. \tag{20}$$

In his section 159, Lamb[12] omitted the final constants in Eqs. (19) and (20), necessary for the continuity of streamfunction at the boundary (18) and for the correct evaluation of the vortex energy.

The energy of the Kirchhoff vortex is proportional to the integral of ψ over the interior of the ellipse; integrating Eq. (20) gives

$$E(a,b) = -\frac{\Gamma^2}{4\pi}\left[\ln\left(\frac{a+b}{2}\right) - \frac{1}{4}\right]. \tag{21}$$

Since the isovortical axisymmetric vortex has radius $\sqrt{ab}$, the energy difference is

$$\Delta E = E(a,b) - E(\sqrt{ab},\sqrt{ab}), \tag{22}$$

which reproduces the result of the general formula (15).

A natural question is whether Eq. (15) can be generalized to isovortical deformations that are more complicated than the elliptical deformation considered here. The answer to this question is negative: our derivation does not generalize to vorticity profiles of the general form $\Omega[|\boldsymbol{f}(\boldsymbol{x})|]$, where $\boldsymbol{f}$ is a nonlinear area-preserving map. More precisely, it can be shown that the factor $\ln[|\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{x}')|/|\boldsymbol{x}-\boldsymbol{x}'|]$, which generalizes Eq. (12), depends on r and r' unless the map $\boldsymbol{f}$ is linear. Thus, for a nonlinear map the integrals equivalent to Eq. (13) are entangled.

It is interesting to note that a formula analogous to Eq. (15) can be found for three-dimensional quasigeostrophic vortices. In this case, the energy $E(a,b,c)$ of an ellipsoidal vortex with semiaxes (a,b,c) is related to the energy of the isovortical spherical vortex by

$$E(a,b,c) = \frac{\sqrt[3]{abc}}{2}\int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \times E(\sqrt[3]{abc},\sqrt[3]{abc},\sqrt[3]{abc}). \tag{23}$$

This formula can been obtained from closely related results in the theory of gravitational potential[13,14] or derived using a change of variables generalizing Eq. (10). A key difference between the two- and three-dimensional formulas is that the energy of the deformed vortex is related to that of the undeformed symmetric vortex by an additive constant in two dimensions and by a multiplicative constant in three dimensions. The additive relationship is rather special: a multiplicative relationship holds for all the two-dimensional systems with vorticity-streamfunction relation[15] of the form $(-\nabla^2)^\alpha\psi=\omega$ provided that $\alpha \neq 1$.

We conclude this letter by noting that the change in vortex energy induced by elliptical distortion of a vortex is a crucial ingredient in physical explanations of the inverse energy cascade of two-dimensional turbulence.[3,10,16–19] The result (14) can be used to construct a simple cartoon of the inverse cascade of freely evolving two-dimensional turbulence. Imagine preparing an initial condition consisting of $N \gg 1$ well separated axisymmetric vortices in a large but finite box. This initial condition is constructed so that there are no correlations between the vortex positions, and the sum of the N nonzero vortex circulations is zero, i.e., there is no net vorticity in the box. Because the vortices are well separated, each vortex will be subject to a local strain due to the other $N-1$ distant vortices, and therefore each vortex is elliptically distorted and loses energy according to Eq. (14). Because the total energy in the box is constant, the vortices must change their configuration so that the loss of self-energy is balanced by an increase in the configuration energy, requiring segregation into clusters of like-signed vortices. This is just a cartoon, of course: it ignores many details, such as the adjustment of an elliptically perturbed vortex,[1,5–8] the nonlinear nature of the strain, and the merging of vortices. Nonetheless, we feel that it offers a useful insight into the physical basis of the inverse energy cascade.

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