SIO203B/MAE294B Final 2015

No computers, calculators, iphones etc.

Problem 1

Find the leading-order, $x \to \infty$, asymptotic expansion of

$$E(x) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{-xt-t^4/4}\,\mathrm{d}t\,, \qquad \text{and} \qquad F(x) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{xt-t^4/4}\,\mathrm{d}t\,. \tag{1}$$

Problem 2

Use multiple scale theory to find an approximate solution of the initial value problem

$$u_{tt} + u = 2\left[\cos(\epsilon t) + \epsilon u^2\right]\,, \qquad \text{with ICs} \qquad u(0) = 0\,, \qquad u_t(0) = 0\,. \tag{2}$$

Problem 3

The function $y(x, \epsilon)$ satisfies

$$\epsilon y'' + \sqrt{x}y' + \mathrm{e}^{-y} = 0\,, \qquad \text{in } 0 < x < 1, \tag{3}$$

and is subject to the BCs $y(0) = 0$ and $y(1) = 1$. *(i)* Find the rescaling for the boundary layer near $x = 0$, and obtain the leading-order inner approximation. *(ii)* Find the leading-order outer approximation and match the two approximations. *(iii)* Construct a uniform approximation. All definite integrals should be evaluated in terms of $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$.

Problem 4

If $\epsilon = 0$ the eigenproblem

$$(1 + \epsilon y)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0\,, \qquad y(0) = y(\pi) = 0\,, \tag{4}$$

has the solution $\lambda = 1$ and $y(x) = a \sin x$. Use perturbation theory ($0 < \epsilon \ll 1$) to investigate the dependence of the eigenvalue $\lambda(\epsilon)$ on a and ϵ. To check your answer, show that if $\epsilon = 1/7$ and $a = 1$ then $\lambda \approx 37/33$.

Problem 5

Find a leading-order $\epsilon \to 0$ approximation to

$$F(\epsilon) \stackrel{\text{def}}{=} \int_0^\infty \frac{x\,\mathrm{d}x}{(\epsilon^2 + x^2)\sqrt{1 + x^2}}\,. \tag{5}$$

Some of these indefinite integrals

$$\int \frac{\mathrm{d}t}{1 + t^2} = \arctan t\,, \qquad \int \frac{\mathrm{d}t}{\sqrt{1 + t^2}} = \ln\left(t + \sqrt{1 + t^2}\right)\,, \tag{6}$$

$$\int \frac{\mathrm{d}t}{t^2\sqrt{1 + t^2}} = -\frac{\sqrt{1 + t^2}}{t}\,, \qquad \int \frac{\mathrm{d}t}{t\sqrt{1 + t^2}} = \ln\left(\frac{t}{1 + \sqrt{1 + t^2}}\right)\,, \tag{7}$$

$$\int \frac{2t\,\mathrm{d}t}{1 + t^2} = \ln\left(1 + t^2\right)\,, \qquad \int \frac{t\,\mathrm{d}t}{\sqrt{1 + t^2}} = \sqrt{1 + t^2}\,. \tag{8}$$

might be useful.

TURN THE PAGE — THERE IS ANOTHER QUESTION

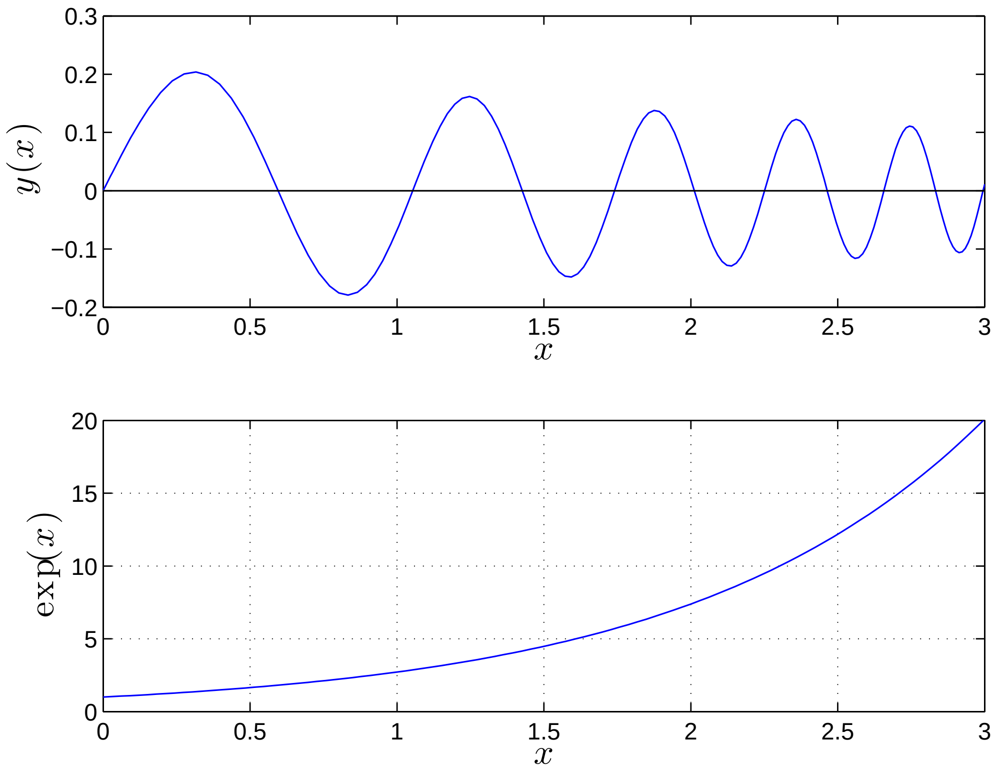

Figure 1: Figure for problem **6**.

Problem 6

The top panel of figure 1 shows the solution to one of the four initial value problems:

$$\begin{aligned}
\epsilon^2 y_1'' - \mathrm{e}^{-x} y_1 = 0\,, \quad y_1(0) = 0\,, \quad y_1'(0) = 1\,,\\
\epsilon^2 y_2'' - \mathrm{e}^{x} y_2 = 0\,, \quad y_2(0) = 0\,, \quad y_2'(0) = 1\,,\\
\epsilon^2 y_3'' + \mathrm{e}^{-x} y_3 = 0\,, \quad y_3(0) = 0\,, \quad y_3'(0) = 1\,,\\
\epsilon^2 y_4'' + \mathrm{e}^{x} y_4 = 0\,, \quad y_4(0) = 0\,, \quad y_4'(0) = 1\,.
\end{aligned}$$

(i) Which $y_n(x)$ is shown in the top panel figure 1? Lucky guesses don't count — explain your answer. *(ii)* Use the WKB approximation and the information in the bottom panel of Figure 1 to estimate the value of ϵ as either a decimal with one significant figure or as a simple fraction.