SIO203B/MAE294B Final 2016

This exam is open notes, but no computers, iPhones or electronic assistance.

Problem 1

The function $y(x, \epsilon)$ satisfies

$$\epsilon y'' + x^{1/7} y' + x^{2/7} y = 0\,, \qquad \text{in } 0 < x < 1, \tag{1}$$

and is subject to the BCs $y(0) = 0$ and $y(1) = 1$. *(i)* Find the rescaling for the boundary layer near $x = 0$, and obtain the leading-order boundary-layer approximation. *(ii)* Find the leading-order outer approximation and match the two approximations. *(iii)* Construct a uniform approximation. All definite integrals should be evaluated in terms of

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t}\, \mathrm{d}t\,. \tag{2}$$

Problem 2

If $\epsilon = 0$ the eigenproblem

$$\mathrm{e}^{\epsilon y} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda y = 0\,, \qquad y(0) = y(\pi) = 0\,, \tag{3}$$

has the solution $\lambda = 1$ and $y(x) = a \sin x$. Use perturbation theory ($\epsilon \to 0$) to investigate the dependence of the eigenvalue $\lambda(\epsilon)$ on a and ϵ. To check your answer, show that if $\epsilon = 1/7$ and $a = 1$ then $\lambda \approx 37/33$. (Use $\pi \approx 22/7$.)

Problem 3

Find a leading order $x \to \infty$ asymptotic approximation to

$$B(x) = \int_0^\pi \mathrm{e}^{\mathrm{i}x(t+\cos t)}\, \mathrm{d}t\,. \tag{4}$$

(There is no need to justify the asymptoticness of the approximation.) You may quote the result

$$\int_0^\infty \mathrm{e}^{\mathrm{i}\beta^p} \mathrm{d}\beta = \Gamma\left(\frac{p+1}{p}\right) \exp\left(\frac{\mathrm{i}\pi}{2p}\right)\,, \qquad \text{provided } p > 1. \tag{5}$$

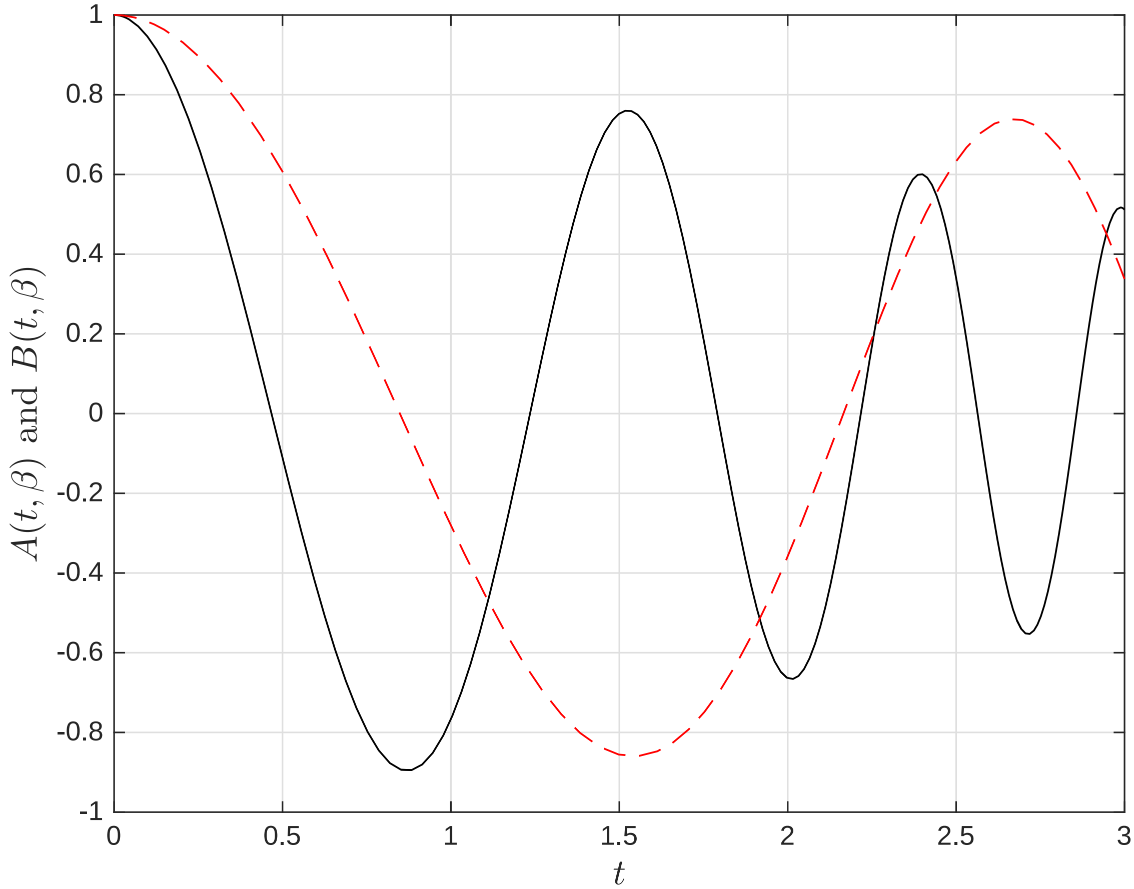

Figure 1: One curve is $A(t)$ and the other is $B(t)$. Which is which?

Problem 4

Dr. Kluge used matlab to solve the initial value problems

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} + \left(\pi\beta^2 + t^2\right)^2 A = 0\,, \qquad A(0) = 1\,, \qquad \frac{\mathrm{d}A}{\mathrm{d}t}(0) = 0\,, \tag{6}$$

and

$$\frac{\mathrm{d}^2 B}{\mathrm{d}t^2} + \left(\pi\beta^2 + t^2\right) B = 0\,, \qquad B(0) = 1\,, \qquad \frac{\mathrm{d}B}{\mathrm{d}t}(0) = 0\,. \tag{7}$$

Kluge produced the numerical solutions in the figure, but has forgotten whether the black solid curve is $A(t,\beta)$ or $B(t,\beta)$. Although Kluge used the same value of the parameter β in both solutions, she has also forgotten the value of β. Help Kluge by telling her whether the black solid curve is $A(t,\beta)$ or $B(t,\beta)$, and estimate β to one significant figure.

Problem 5

Use multiple-scale theory to investigate the parametrically forced oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + (1 + \epsilon \cos t)\, x = 0\,. \tag{8}$$

Show that the amplitude of the oscillation grows exponentially as $\mathrm{e}^{\gamma t}$ and estimate the growth-rate $\gamma(\epsilon)$ as $\epsilon \to 0$.