SIO203B/MAE294B Final 2017

This exam is open notes, but no computers, iPhones or electronic assistance.

Problem 1

Consider the initial value problem:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + w = 2\cos(\epsilon t) + 2\epsilon w^2\,, \qquad \text{with ICs} \qquad w(0) = \frac{\mathrm{d}w}{\mathrm{d}t}(0) = 0\,. \tag{1}$$

Supposing that $\epsilon \ll 1$, use the method of multiple time scales ($s = \epsilon t$) to obtain an approximate solution valid on times of order ϵ^{-1}.

Problem 2

The beta function is

$$B(x,y) \stackrel{\text{def}}{=} \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t\,. \tag{2}$$

With a change of variables show that

$$B(x,y) = \int_0^\infty \mathrm{e}^{-xv}(1-\mathrm{e}^{-v})^{y-1}\,\mathrm{d}v\,. \tag{3}$$

Obtain the leading-order approximation to $B(x,y)$ in the limit $x \to \infty$ with y fixed. All definite integrals should be evaluated in terms of

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t\,. \tag{4}$$

Problem 3

Find a leading order $x \to \infty$ asymptotic approximation to

$$F(x) = \int_0^\pi \cos\left(x\mathrm{e}^{t^2}\right)\,\mathrm{d}t\,. \tag{5}$$

(There is no need to justify the asymptoticness of the approximation.) You may quote the result

$$\int_0^\infty \cos a^2 t^2\,\mathrm{d}t = \int_0^\infty \sin a^2 t^2\,\mathrm{d}t = \frac{1}{2a}\sqrt{\frac{\pi}{2}}\,.$$

Problem 4

The function $y(x,\epsilon)$ satisfies

$$\epsilon y'' + 2xy' + 2xy = 0\,, \qquad \text{in } 0 < x < 1, \tag{6}$$

and is subject to the BCs $y(0) = 0$ and $y(1) = 1$. *(i)* Find the rescaling for the boundary layer near $x = 0$, and obtain the leading-order, $\epsilon \to 0$, boundary-layer approximation. *(ii)* Find the leading-order outer approximation and match the two approximations. *(iii)* Construct a uniform approximation.

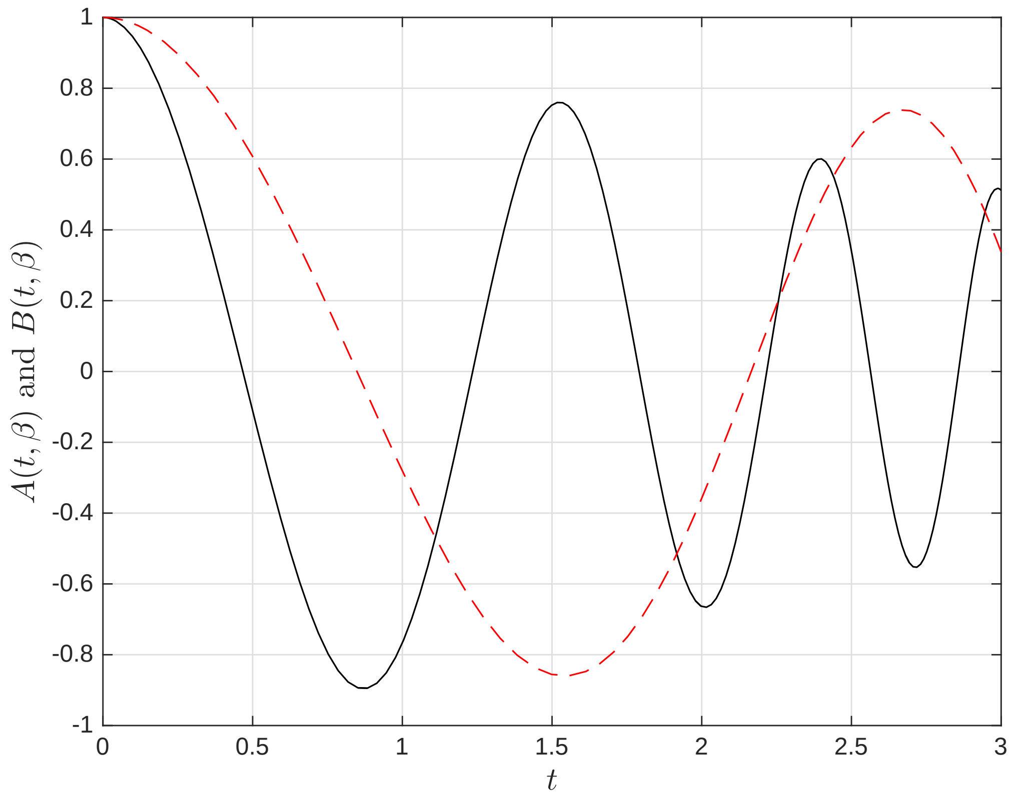

Figure 1: One curve is $A(t)$ and the other is $B(t)$. Which is which?

Problem 5

Dr. Kluge used matlab to solve the initial value problems

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} + \left(\pi\beta^2 + t^2\right)^2 A = 0\,, \qquad A(0) = 1\,, \qquad \frac{\mathrm{d}A}{\mathrm{d}t}(0) = 0\,, \tag{7}$$

and

$$\frac{\mathrm{d}^2 B}{\mathrm{d}t^2} + \left(\pi\beta^2 + t^2\right) B = 0\,, \qquad B(0) = 1\,, \qquad \frac{\mathrm{d}B}{\mathrm{d}t}(0) = 0\,. \tag{8}$$

Kluge produced the numerical solutions in the figure, but has forgotten whether the black solid curve is $A(t,\beta)$ or $B(t,\beta)$. Although Kluge used the same value of the parameter β in both solutions, she has also forgotten the value of β. Help Kluge by telling her whether the black solid curve is $A(t,\beta)$ or $B(t,\beta)$, and estimate β to one significant figure.

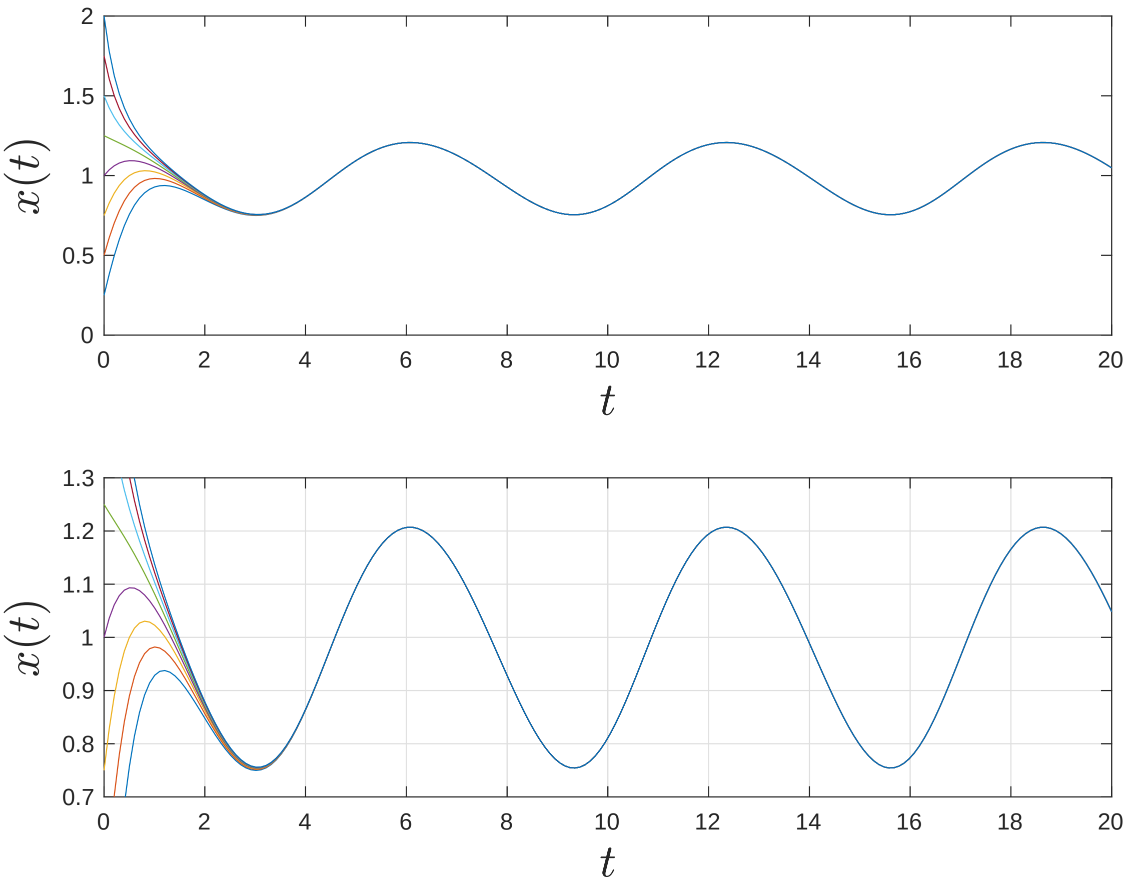

Figure 2: Evolution of 7 initial conditions. The bottom panel is an expanded view showing small oscillations at long time.

Problem 6

Figure 2 shows solutions of the differential equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 1 - p^2 + \epsilon \cos(t + \phi)\,. \tag{9}$$

(i) Estimate the value of the small parameter ϵ used to make the figure. *(ii)* Discuss in quantitative terms the *asymmetry* of the oscillations about $p = 1$ i.e., the oscillation is between about 0.75 and 1.2 and the mean of those two numbers is less than 1.