SIO203B/MAE294B Final 2023

This exam is open notes. The notes can be paper or on an electronic device.

Problem 1

Find a *three*-term $\epsilon \ll 1$ approximation to the four roots of the perturbed quartic polynomial

$$(x - \mathrm{i})^4 + \epsilon \mathrm{e}^{\pi x} = 0\,. \tag{1}$$

Problem 2

The function $y(x, \epsilon)$ satisfies

$$\epsilon y'' + x^{3/4} y' + x^{1/4} y = 0\,, \qquad \text{in } 0 < x < 1. \tag{2}$$

Boundary conditions are $y(0) = 0$ and $y(1) = 1$. *(i)* In the limit $\epsilon \to 0$, find the rescaling for the boundary layer near $x = 0$, and obtain the leading-order boundary-layer approximation. *(ii)* Find the leading-order outer approximation and match the two approximations. *(iii)* Construct a uniform approximation. All definite integrals should be evaluated in terms of

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t}\, \mathrm{d}t\,. \tag{3}$$

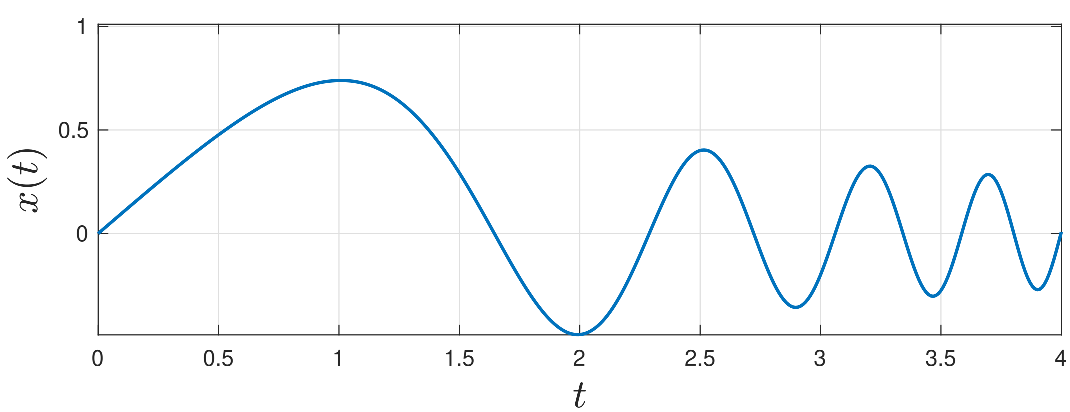

Figure 1: One of Kluge's solutions – but which one?

Problem 3

Dr. Kluge used MATLAB to solve the initial value problems

$$\epsilon^2 u'' + (1 + t^2)^2 u = 0\,, \qquad u(0) = 0\,, \quad u'(0) = 1\,, \tag{4}$$

$$\epsilon^2 v'' - (1 + t^2)^2 v = 0\,, \qquad v(0) = 0\,, \quad v'(0) = 1\,, \tag{5}$$

$$\epsilon^2 w'' + (1 + t^2)^{-2} w = 0\,, \qquad w(0) = 0\,, \quad w'(0) = 1\,. \tag{6}$$

Which solution is shown in figure 1? Estimate the value of ϵ used by Kluge.

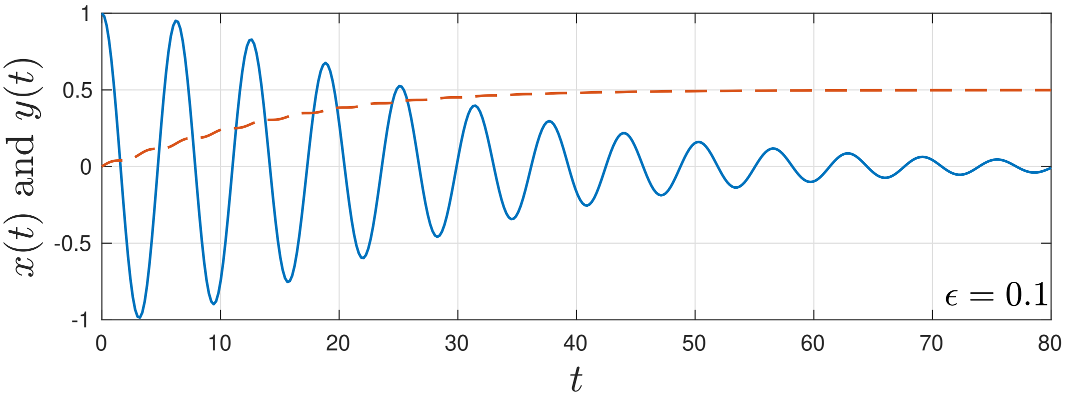

Figure 2: Numerical solution of (7) and (8) with $\epsilon = 1/10$.

Problem 4

Find the leading-order approximation, valid on $t = O(\epsilon^{-1})$, to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\epsilon y \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0\,, \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\epsilon}{2} x^2\,. \tag{7}$$

If the initial conditions are

$$x(0) = 1\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0\,, \qquad y(0) = 0, \tag{8}$$

show that $y \to 1/2$ as $t \to \infty$ (see figure 2).

Problem 5

Consider

$$A(x) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{xt - t^3/3}\,\mathrm{d}t\,. \tag{9}$$

(i) Show that

$$\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} - xA = 1\,. \tag{10}$$

(ii) Find a leading-order asymptotic approximation to $A(x)$ in the limit $x \to -\infty$. *(iii)* Find a leading-order asymptotic approximation to $A(x)$ in the limit $x \to +\infty$.