SIO203B/MAE294B Mid-term 2015

This exam is open notes, but no computers, iPhones or electronic assistance.

Problem 1

Find two terms in the $x \to \infty$ asymptotic expansion of

$$A(x) \stackrel{\text{def}}{=} \int_0^x \mathrm{e}^{-t^3}\,\mathrm{d}t\,. \tag{1}$$

Express all integrals in your answer in terms of $\Gamma(z) \equiv \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$. Justify the asymptoticness of your two-term expansion by showing that remainder is negligible relative to the second term as $x \to \infty$.

Problem 2

Find the leading-order, $x \to \infty$, asymptotic expansion of

$$R(x) \stackrel{\text{def}}{=} \int_0^\infty \exp\left(-xt - \mathrm{e}^t\right)\,\mathrm{d}t\,, \qquad \text{and} \qquad S(x) \stackrel{\text{def}}{=} \int_0^\infty \exp\left(xt - \mathrm{e}^t\right)\,\mathrm{d}t\,. \tag{2}$$

Problem 3

Find a two-term $\epsilon \to 0$ approximation to all roots of the polynomial $y^3 - y^2 + \epsilon = 0$.

Problem 4

$x(t)$ is defined via the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \exp\left(\frac{x}{10}\right) - x\,, \qquad \text{with IC} \qquad x(0) = 0\,. \tag{3}$$

Find $\lim_{t\to\infty} x(t)$ to three significant figures.

Problem 5

(i) Use multiple scale theory to find an approximate solution of

$$u_{tt} + u = \mathrm{e}^{\epsilon t} + \epsilon \mathrm{e}^{-\epsilon t} u^2\,, \qquad \text{with ICs} \qquad u(0) = u_t(0) = 0\,, \tag{4}$$

valid on the long time scale $t \sim \epsilon^{-1}$. *(ii)* Consider

$$v_{tt} + v = u\,, \qquad \text{with ICs} \qquad v(0) = v_t(0) = 0\,. \tag{5}$$

On the right of (5), $u(t,\epsilon)$ is the solution from part *(i)*. Find a leading-order approximation to $v(t,\epsilon)$ valid on the long time scale $t \sim \epsilon^{-1}$.

Hint: The forcing, $u(t,\epsilon)$, on the right of the oscillator equation (5) is nearly resonant. So you should anticipate that $v(t,\epsilon)$ is much larger than $u(t,\epsilon)$.