SIO203B/MAE294B Mid-term 2023

This exam is open notes. The notes can be paper or on an electronic device.
All problems have equal weight.

Problem 1

Find a two-term approximation to all roots of the quartic polynomial

$$(x-1)^4 - \epsilon x^3 = 0\,. \tag{1}$$

Problem 2

Find the leading-order $x \to \infty$ behaviour of

$$E_2(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t^2}\mathrm{d}t\,. \tag{2}$$

You should justify neglect of the remainder in the limit $x \to \infty$.

Problem 3

Consider

$$-h_x = \epsilon\left(\ln\frac{1}{h}\right)_{xx} + 2x\,, \tag{3}$$

posed on the interval $\epsilon < x < 1$ with boundary conditions $h(0) = 1/7$ and $h(1) = 0$. Find the leading-order $\epsilon \to 0$ solution in both the interior and the boundary layer and construct a uniform solution.

Turn the page for problem 4

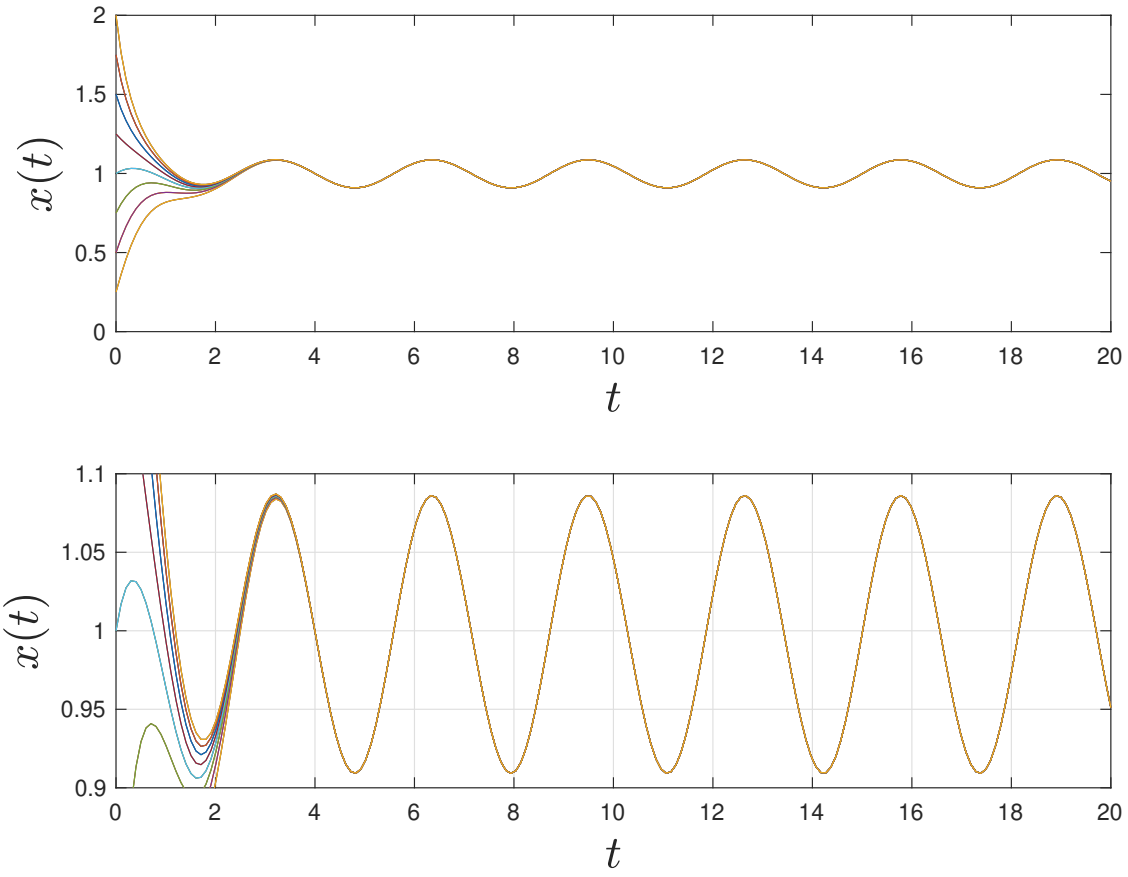

Figure 1: Evolution of 7 initial conditions. The bottom panel is an expanded view showing small oscillations about $x = 1$ at long time.

Problem 4

Figure 1 shows solutions of one of the following differential equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 1 - p^2 + \epsilon \cos t\,, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = 1 - q^2 + \epsilon \cos 2t\,, \tag{4}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -1 + r^2 + \epsilon \cos t\,, \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = -1 + s^2 + \epsilon \cos 2t\,. \tag{5}$$

The parameter ϵ is a smallish positive number. *(i)* Without explicitly solving a differential equation, determine which equation has been solved to produce figure 1? *(ii)* Estimate the value of ϵ used to make the figure. (A rough numerical answer is required – you can leave it as a simple fraction. A simple fraction is the ratio of two integers e.g. 17/9 or 1/3.)