Assignment 2 SIO203B/MAE294B, 2025

Due by mid-night Wednesday April 16th
Submit by email to `wryoung@ucsd.edu`
with subject line `Second asymptotology assignment`

Belligerent drunks again

Let's make a small change to the formulation of the belligerent-drunks example in section **3.2** of the notes. Recall that the steady state density equation is

$$\kappa u_{xx} - \mu u^2 = 0 \,. \tag{1}$$

Suppose that we model the bars using a Neumann boundary condition. This means that the flux of drunks, rather than the concentration, is prescribed at $x = 0$ and ℓ:

$$\kappa u_x(0,t) = -F \,, \qquad \text{and} \qquad \kappa u_x(\ell,t) = F \,, \tag{2}$$

where F, with dimensions drunks per second, is the flux entering the domain from the bars. Try to repeat *all calculations* in section **3.2** including the analog of the weakly interacting limit $\alpha \ll 1$ perturbation expansion (find a "reasonable" number of terms). You'll find that it is not straightforward and that a certain amount of ingenuity is required to understand the weakly interacting limit with fixed-flux boundary conditions.

Hint: the difficulty is not that the problem above is nonlinear. So if you're absolutely stuck you can retreat to an easier linear problem that poses the same challenge:

$$\kappa v_{xx} - \nu v = 0 \,, \tag{3}$$

with Neumann BCs

$$\kappa v_x(0,t) = -F \,, \qquad \text{and} \qquad \kappa v_x(\ell,t) = F \,. \tag{4}$$

Solve this v-problem exactly and then analyze the solution with $\ell^2\nu/\kappa \ll 1$ to understand the scaling.

A logistic equation with time varying carrying capacity

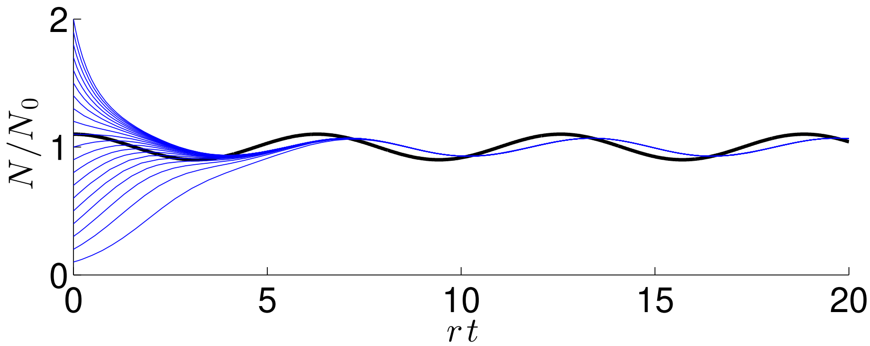

Figure 1: Numerical solution of (5) with various initial conditions. The black sinusoid is the periodic-in-time carrying capacity. At large time all initial conditions convergence to a periodic solution that lags the carrying capacity.

Consider the logistic equation with a periodically varying carrying capacity:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \qquad \text{with} \qquad K(t) = K_0 + K_1 \cos \omega t\,. \tag{5}$$

The initial condition is $N(0) = M$. *(i)* Based on the $K_1 = 0$ solution, non-dimensionalize this problem. Show that there are three non-dimensional control parameters. *(ii)* Suppose that K_1 is a perturbation i.e., $K_1/K_0 \ll 1$. The numerical solution in Figure 1 shows that eventually the initial condition is "forgotten" and all solutions converge to a periodic oscillation about the mean carrying capacity K_0. Use perturbation theory to determine the amplitude and phase of the long-term oscillation. ("Long-term" means the large-time solution. You do not have to solve the complete initial value problem.) Show that peak population lags peak carrying capacity.

An integral

Consider

$$J(x,p,q) \stackrel{\text{def}}{=} \int_x^\infty \mathrm{e}^{-pt^q}\,\mathrm{d}t\,. \tag{6}$$

(a) Explain why this problem makes sense only if p and q are both greater than zero (twelve words or less). (b) With a change of variable express $J(x,p,q)$ in terms of the simpler function $K(x,q) = J(x,1,q)$. (c) Find the leading-order $x \to \infty$ approximation to $K(x,q)$. (There is no need to justify asymptoticness – I'll assume that discussion in class, and in the recitation, has been sufficient.)