Second recitation SIO203B/MAE294B, 2025

For discussion in the recitation on Friday April 11th

Some of these problems will be set for hand-in

Problem 2.13 with $a = 1$

Read section 2.5 of the notes and use the L_1 and L_2 notation found there. Consider

$$\epsilon y = \mathrm{e}^{-y} . \tag{1}$$

Use iteration to find a two or three terms in the $\epsilon \to 0$ asymptotic solution of (1). Use MATLAB to make a graphical comparison between your approximation and the answer.

Problem 3.1 Air resistance

(i) Consider the projectile problem with linear drag:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \mu \frac{\mathrm{d}z}{\mathrm{d}t} = -g_0 , \tag{2}$$

and the initial conditions $z(0) = 0$ and $\mathrm{d}z/\mathrm{d}t = u$. Find the solution with no drag, $\mu = 0$, and calculate the time aloft, τ. *(ii)* Suppose that the drag is small – make this precise by non-dimensionalizing the equation of motion and exhibiting the relevant small parameter ϵ. Hint: non-dimensionalize so that $(g_0, u) \mapsto (1, 1)$. *(iii)* Use a RPS to determine the first correction to τ associated with non-zero drag. *(iv)* Find the time to reach maximum altitude. Does the projectile take longer going up or coming down? *(v)* Integrate the non-dimensional differential equation exactly and obtain a transcendental equation for $\tau(\epsilon)$. Solve this transcendental equation approximately in the limit $\epsilon \to 0$. Make sure the $\epsilon \to 0$ solution agrees with the earlier RPS.

Problem 4.1 with $q = 3$

Consider

$$A(x, p) \stackrel{\text{def}}{=} \int_x^\infty \mathrm{e}^{-pt^3} \, \mathrm{d}t . \tag{3}$$

With a change of variable express $A(x, p)$ in terms of the simpler function $B(x) = A(x, 1)$. Find the leading-order $x \to \infty$ approximation to $B(x)$.

Iterating belligerent drunks

Read section 3.2 so that you understand where the boundary value problem

$$u_{xx} = \alpha u^2 , \qquad u(\pm 1) = 1 \tag{4}$$

comes from. Consider the $\alpha \ll 1$ iterative schemes

$$(a) \qquad u^{(0)} = 1 , \qquad u_{xx}^{(n+1)} = \alpha {u^{(n)}}^2 , \tag{5}$$

and

$$(a) \qquad u^{(0)} = 1 , \qquad u_{xx}^{(n+1)} = \alpha u^{(n)} u^{(n+1)} \tag{6}$$

Calculate $u^{(1)}(x)$ in both cases and compare with the results in the lecture notes. Which scheme is likely to be more accurate? Discuss the difficulty of proceeding to $u^{(2)}(x)$.

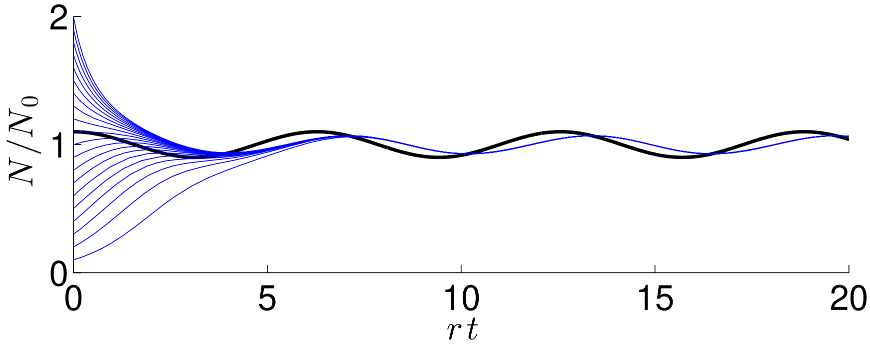

Figure 1: Numerical solution of (7) with various initial conditions. The carrying capacity, $K(t)$, is the heavy black curve. At large time all initial conditions convergence to a periodic solution that lags the carrying capacity i.e. the peak population is is after the peak carrying capacity.

A logistic equation with time varying carrying capacity

Consider the logistic equation with a periodically varying carrying capacity:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \qquad \text{with} \qquad K(t) = K_0 + K_1 \cos \omega t\,. \tag{7}$$

The initial condition is $N(0) = N_0$. *(i)* Based on the $K_1 = 0$ solution, non-dimensionalize this problem. Show that there are three control parameters. *(ii)* Suppose that K_1 is a perturbation i.e., $\epsilon \stackrel{\text{def}}{=} K_1/K_0 \ll 1$ and that $N(t) \approx K_0$. Use first-order terturbation theory to find the periodic-in-time solution of the perturbed problem e.g. see Figure 1. *(iii)* How does the phase lag between the population, $N(t)$, and the carrying capacity $K(t)$ depend on parameters?

Belligerent drunks with Neumann boundary conditions

This problem is difficult – finding the "best" way to non-dimensionalize the problem is tricky. Don't spend a lot of time on this at the expense of the other problems.

Let's make a small change to the formulation of the belligerent-drunks example in section 4.2 of the notes. Suppose that we model the bars at $x = 0$ and ℓ using a Neumann boundary condition. This means that the flux of drunks, rather than the concentration, is prescribed at $x = 0$ and ℓ. Thus the boundary condition in the notes is changed to

$$\kappa u_x(0,t) = -F\,, \qquad \text{and} \qquad \kappa u_x(\ell,t) = F\,, \tag{8}$$

where the constant F, with dimensions drunks per second, is the flux entering the domain from the bars. Try to repeat *all calculations* in section 4.2, including the analog of the $\beta \ll 1$ perturbation expansion. You'll find that it is not straightforward and some ingenuity is required to understand the weakly interacting limit with fixed-flux boundary conditions.