Fourth recitation SIO203B/MAE294B, 2025

For discussion in the recitation on Friday April 25th

Basic BL problem

(i) Find a leading order uniformly valid solution of

$$-h_x = \epsilon h_{xx} + x\,, \qquad h(0) = h(1) = 0\,. \tag{1}$$

(ii) Solve the BVP above exactly and compare the exact solution to the boundary layer approximation with $\epsilon = 0.1$.

A not so basic BL problem

Find the leading-order uniformly valid boundary-layer solution to the Stommel problem

$$-\left(\mathrm{e}^x g\right)_x = \epsilon g_{xx} + 1\,, \qquad \text{with BCs} \quad g(0) = g(1) = 0 \;. \tag{2}$$

Do the same for

$$\left(\mathrm{e}^x f\right)_x = \epsilon f_{xx} + 1\,, \qquad \text{with BCs} \quad f(0) = f(1) = 0\,. \tag{3}$$

A nonlinear BL problem

Use boundary layer theory to find leading order solution of

$$h_x = \epsilon \left(\tfrac{1}{3} h^3\right)_{xx} + 1\,, \tag{4}$$

on the domain $0 < x < 1$ with boundary conditions $h(0) = h(1) = 0$. You can check your answer by showing that $h = 1/2$ at $x \approx 1 - (\ln 2 - 5/8)\epsilon$.

Asymptotic notation

True or false as $x \to \infty$

$$(i)\ x + \frac{1}{x} \overset{?}{\sim} x\,, \qquad (ii)\ x + \sqrt{x} \overset{?}{\sim} x\,, \qquad (iii)\ \exp\left(x + \frac{1}{x}\right) \overset{?}{\sim} \exp(x)\,, \tag{5}$$

$$(iv)\ \exp\left(x + \sqrt{x}\right) \overset{?}{\sim} \exp(x)\,, \qquad (v)\ \cos\left(x + \frac{1}{x}\right) \overset{?}{\sim} \cos x\,, \qquad (v)\ \frac{1}{x} \overset{?}{\sim} 0\,? \tag{6}$$

A corrugated boundary problem

Consider the partial differential equation

$$\kappa \left(C_{xx} + C_{zz}\right) - \mu C = 0 \tag{7}$$

in the region above $z = h(x)$, with $h(x) = a\cos kx$. The boundary conditions are $C(x, a\cos kx) = C_*$ and $C(x,z) \to 0$ as $z \to \infty$. *(i)* Describe a physical situation governed by this boundary value problem. *(ii)* Solve the problem with $a = 0$. *(iii)* Based on your exact solution, non-dimensionalize the problem with non-zero a and determine the non-dimensional control parameters. *(iv)* Use perturbation theory to find the first effects of small non-zero a on the "inventory"

$$A \overset{\text{def}}{=} \frac{k}{2\pi} \int_{h(x)}^{\infty} \int_0^{2\pi/k} C(x,z)\, \mathrm{d}x\, \mathrm{d}z\,. \tag{8}$$

(I think you'll have to go to second order in the relevant small parameter.)

An elementary boundary-layer example

In the lectures we saw that the exact solution of

$$\epsilon^2 v'' - v = 0\,, \qquad \text{and} \qquad v(\pm 1) = 1\,, \tag{9}$$

is

$$v = \frac{\cosh(x/\epsilon)}{\cosh(1/\epsilon)}\,. \tag{10}$$

(i) Draw a graph of the solution with $\epsilon = 1$, $1/4$, and $1/16$. In the limit $\epsilon \to 0$ simplify the solution in (10) in the neighbourhood of $x = -1$. Hint: hold a magnifying glass to this region by introducing the "stretched coordinate"

$$X \stackrel{\text{def}}{=} \frac{x+1}{\epsilon}\,, \qquad \Rightarrow \qquad x = -1 + \epsilon X\,. \tag{11}$$

(ii) Derive the simplified version of $v(x, \epsilon)$ by direct solution of (9) in vicinity of $x = -1$.

Another boundary-layer problem

Find an approximate solution of the boundary value problem:

$$10^{-12} w_{xx} = \mathrm{e}^{x^4}\, w\,, \qquad \text{with BCs} \qquad v(\pm 1) = 1\,. \tag{12}$$

The last quadratic equation problem

By inspection find a simple approximation to the solutions of the quadratic equation $x^2 - 19x + 1 = 0$. How about $x^3 - 19x + 1 = 0$? How about $x^{n+1} - 19x + 1 = 0$ as $n \to \infty$?

An interesting integral

Easy: find the leading order $x \to \infty$ asymptotic expansion of

$$F(x) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{-xt - t^4/4}\, \mathrm{d}t\,. \tag{13}$$

More difficult: find the leading order $x \to -\infty$ asymptotic expansion.