equations (e.g., the primitive equations) may be written in non-dimensional form, as

$$\frac{\partial \phi}{\partial t} = F(\phi, \epsilon) \tag{5.92}$$

where ϕ is a set of variables, F is some operator and ϵ is a small parameter, such as the Rossby number. Suppose also that this set of equations has various invariants (such as energy and potential vorticity) that hold for any value of ϵ. The asymptotically derived lowest-order model (such as quasi-geostrophy) is simply a version of this equation set valid in the limit $\epsilon = 0$, and therefore it will preserve the invariants of the original set. These invariants may seem to have a different form in the simplified set: for example, in deriving the hydrostatic primitive equations from the Navier–Stokes equations the small parameter is the aspect ratio, and this multiplies the vertical velocity. Thus, in the limit of zero aspect ratio, and therefore in the primitive equations, the kinetic energy component of the energy invariant has contributions only from the horizontal velocity. In other cases, some invariants may be reduced to trivialities in the simplified set. On the other hand, there is nothing to preclude new invariants emerging that hold only in the limit $\epsilon = 0$, and enstrophy (considered later in this chapter) is one example.

5.4 THE CONTINUOUSLY STRATIFIED QUASI-GEOSTROPHIC SYSTEM

We now consider the quasi-geostrophic equations for the continuously stratified hydrostatic system. The primitive equations of motion are given by (5.15), and we extract the mean stratification so that the thermodynamic equation is given by (5.17). We also stay on the β-plane for simplicity. Readers who wish for a briefer, more informal derivation may peruse the box on page 218; however, it is important to realize that there is a systematic asymptotic derivation of the quasi-geostrophic equations, for it is this that ensures that the resulting equations have good conservation properties, as explained in section 5.3.3.

5.4.1 Scaling and assumptions

The scaling assumptions we make are just those we made for the shallow water system on page 203, with a deformation radius now given by $L_d = NH/f_0$. The non-dimensionalization and scaling are initially precisely that of section 5.1.2, and we obtain the following non-dimensional equations:

horizontal momentum: $$Ro\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} + \hat{\boldsymbol{f}} \times \hat{\boldsymbol{u}} = -\nabla_z \hat{\phi}, \tag{5.93}$$

hydrostatic: $$\frac{\partial \hat{\phi}}{\partial \hat{z}} = \hat{b}, \tag{5.94}$$

mass continuity: $$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{1}{\tilde{\rho}}\frac{\partial \tilde{\rho}\hat{w}}{\partial \hat{z}} = 0, \tag{5.95}$$

thermodynamic: $$Ro\frac{\mathrm{D}\hat{b}}{\mathrm{D}\hat{t}} + \left(\frac{L_d}{L}\right)^2 \hat{w} = 0. \tag{5.96}$$

In Cartesian coordinates we may express the Coriolis parameter as

$$\boldsymbol{f} = \boldsymbol{f}_0 + \beta y\, \mathbf{k}, \tag{5.97}$$

where $\boldsymbol{f}_0 = f_0 \mathbf{k}$. The variation of the Coriolis parameter is assumed to be small (this is a key difference between the quasi-geostrophic system and the planetary-geostrophic system), and in particular we shall assume that βy is approximately the size of the relative vorticity, and so is much smaller than f_0 itself.[6] Thus,

$$\beta y \sim \frac{U}{L}, \qquad \beta \sim \frac{U}{L^2}, \tag{5.98}$$

and so we define an $\mathcal{O}(1)$ non-dimensional beta parameter by

$$\widehat{\beta} = \frac{\beta L^2}{U} = \frac{\beta L}{Ro f_0}. \tag{5.99}$$

From this it follows that if $f = f_0 + \beta y$, the corresponding non-dimensional version is

$$\widehat{f} = \widehat{f}_0 + Ro\,\widehat{\beta}\widehat{y}. \tag{5.100}$$

where $\widehat{f} = f/f_0$ and $\widehat{f}_0 = f_0/f_0 = 1$.

5.4.2 Asymptotics

We now expand the non-dimensional dependent variables in an asymptotic series in Rossby number, and write

$$\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}_0 + Ro\,\widehat{\boldsymbol{u}}_1 + \cdots, \qquad \widehat{\phi} = \widehat{\phi}_0 + Ro\,\widehat{\phi}_1 + \cdots, \qquad \widehat{b} = \widehat{b}_0 + Ro\,\widehat{b}_1 + \cdots. \tag{5.101}$$

Substituting these into the equations of motion, the lowest-order momentum equation is simply geostrophic balance,

$$\widehat{\boldsymbol{f}}_0 \times \widehat{\boldsymbol{u}}_0 = -\nabla\widehat{\phi}_0 \tag{5.102}$$

with a *constant* value of the Coriolis parameter. (For the rest of this chapter we drop the subscript z from the ∇ operator.) From (5.102) it is evident that

$$\nabla\cdot\widehat{\boldsymbol{u}}_0 = 0. \tag{5.103}$$

Thus, the horizontal flow is, to leading order, non-divergent; this is a consequence of geostrophic balance, and is *not* a mass conservation equation. Using (5.103) in the mass conservation equation, (5.95), gives

$$\frac{\partial}{\partial\widehat{z}}(\widetilde{\rho}\widehat{w}_0) = 0, \tag{5.104}$$

which implies that if w_0 is zero somewhere (e.g., at a solid surface) then w_0 is zero everywhere (essentially the Taylor–Proudman effect). A physical way of saying this is that the scaling estimate $W = UH/L$ is an overestimate of the size of the vertical velocity, because even though $\partial w/\partial z \approx -\nabla\cdot\boldsymbol{u}$, the horizontal divergence of the geostrophic flow is small if f is nearly constant and $|\nabla\cdot\boldsymbol{u}| \ll U/L$. We might have anticipated this from the outset, and scaled w differently, perhaps using the geostrophic vorticity balance estimate, $w \sim \beta UH/f_0 = Ro\,UH/L$ as the scaling factor for w, but there is no a priori guarantee that this would be correct.

At next order the momentum equation is

$$\frac{D_0\hat{\boldsymbol{u}}_0}{D\hat{t}} + \hat{\beta}\hat{y}\mathbf{k}\times\hat{\boldsymbol{u}}_0 + \hat{\boldsymbol{f}}\times\hat{\boldsymbol{u}}_1 = -\nabla\hat{\phi}_1, \tag{5.105}$$

where $D_0/Dt = \partial/\partial\hat{t} + (\hat{\boldsymbol{u}}_0\cdot\nabla)$, and the next order mass conservation equation is

$$\nabla_z\cdot(\tilde{\rho}\hat{\boldsymbol{u}}_1) + \frac{\partial}{\partial z}(\tilde{\rho}\hat{w}_1) = 0. \tag{5.106}$$

From (5.96), the lowest-order thermodynamic equation is just

$$\left(\frac{L_d}{L}\right)^2 \hat{w}_0 = 0 \tag{5.107}$$

provided that, as we have assumed, the scales of motion are not sufficiently large that $Ro(L/L_d)^2 = \mathcal{O}(1)$. (This is a key difference between quasi-geostrophy and planetary geostrophy.) At next order we obtain an evolution equation for the buoyancy, and this is

$$\frac{D_0\hat{b}_0}{D\hat{t}} + \hat{w}_1\left(\frac{L_d}{L}\right)^2 = 0. \tag{5.108}$$

The potential vorticity equation

To obtain a single evolution equation for lowest-order quantities we eliminate w_1 between the thermodynamic and momentum equations. Cross-differentiating the first-order momentum equation (5.105) gives the vorticity equation,

$$\frac{\partial\hat{\zeta}_0}{\partial\hat{t}} + (\hat{\boldsymbol{u}}_0\cdot\nabla)\hat{\zeta}_0 + \hat{v}_0\hat{\beta} = -\hat{f}_0\nabla_z\cdot\hat{u}_1. \tag{5.109}$$

(In dimensional terms, the divergence on the right-hand side is small, but is multiplied by the large term f_0, and their product is of the same order as the terms on the left-hand side.) Using the mass conservation equation (5.106), (5.109) becomes

$$\frac{D_0}{D\hat{t}}(\zeta_0 + \hat{f}) = \frac{\hat{f}_0}{\tilde{\rho}}\frac{\partial}{\partial z}(w_1\tilde{\rho}). \tag{5.110}$$

Combining (5.110) and (5.108) gives

$$\frac{D_0}{D\hat{t}}(\zeta_0 + \hat{f}) = -\frac{\hat{f}_0}{\tilde{\rho}}\frac{\partial}{\partial\hat{z}}\left[\frac{D_0}{D\hat{t}}(F\tilde{\rho}\hat{b}_0)\right], \tag{5.111}$$

where $F \equiv (L/L_d)^2$. The right-hand side of this equation is

$$\frac{\partial}{\partial\hat{z}}\left(\frac{D_0\hat{b}_0}{D\hat{t}}\right) = \frac{D_0}{D\hat{t}}\left(\frac{\partial\hat{b}_0}{\partial\hat{z}}\right) + \frac{\partial\hat{\boldsymbol{u}}_0}{\partial\hat{z}}\cdot\nabla\hat{b}_0. \tag{5.112}$$

The second term on the right-hand side vanishes identically using the thermal wind equation

$$\mathbf{k}\times\frac{\partial\hat{\boldsymbol{u}}_0}{\partial\hat{z}} = -\frac{1}{\hat{f}_0}\nabla\hat{b}_0, \tag{5.113}$$

and so (5.111) becomes

$$\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}\left[\hat{\zeta}_0 + \hat{f} + \frac{\hat{f}_0}{\tilde{\rho}}\frac{\partial}{\partial \hat{z}}\left(\tilde{\rho}F\hat{b}_0\right)\right] = 0, \tag{5.114}$$

or, after using the hydrostatic equation,

$$\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}\left[\hat{\zeta}_0 + \hat{f} + \frac{\hat{f}_0}{\tilde{\rho}}\frac{\partial}{\partial \hat{z}}\left(\tilde{\rho}F\frac{\partial\hat{\phi}_0}{\partial z}\right)\right] = 0. \tag{5.115}$$

Since the lowest-order horizontal velocity is divergence-free, we can define a streamfunction $\hat{\psi}$ such that

$$\hat{u}_0 = -\frac{\partial\hat{\psi}}{\partial\hat{y}}, \qquad \hat{v}_0 = \frac{\partial\hat{\psi}}{\partial\hat{x}} \tag{5.116}$$

where also, using (5.102), $\phi_0 = \hat{f}_0\hat{\psi}$. The vorticity is then given by $\hat{\zeta}_0 = \nabla^2\hat{\psi}$ and (5.115) becomes a single equation in a single unknown, to wit

$$\boxed{\frac{\mathrm{D}_0}{\mathrm{D}\hat{t}}\left[\nabla^2\hat{\psi} + \hat{\beta}\hat{y} + \frac{\hat{f}_0^2}{\tilde{\rho}}\frac{\partial}{\partial\hat{z}}\left(\tilde{\rho}F\frac{\partial\hat{\psi}}{\partial\hat{z}}\right)\right] = 0}\,, \tag{5.117}$$

where the material derivative is evaluated using $\hat{\boldsymbol{u}}_0 = \mathbf{k}\times\nabla\hat{\psi}$. This is the non-dimensional form of the quasi-geostrophic potential vorticity equation, one of the most important equations in dynamical meteorology and oceanography. In deriving it we have reduced the Navier–Stokes equations, which are six coupled nonlinear partial differential equations in six unknowns (u, v, w, T, p, ρ) to a single (albeit nonlinear) first-order partial differential equation in a single unknown.[7]

Dimensional equations

The dimensional version of the quasi-geostrophic potential vorticity equation may be written as

$$\boxed{\begin{gathered}\frac{\mathrm{D}q}{\mathrm{D}t} = 0,\\ q = \nabla^2\psi + f + \frac{f_0^2}{\tilde{\rho}}\frac{\partial}{\partial z}\left(\frac{\tilde{\rho}}{N^2}\frac{\partial\psi}{\partial z}\right)\end{gathered}}\,, \tag{5.118a,b}$$

where only the variable part of f (e.g., βy) is relevant in the second term on the right-hand side of the expression for q. The quantity q is known as the *quasi-geostrophic potential vorticity*. It is analogous to the exact (Ertel) potential vorticity (see section 5.5 for more about this), and it is conserved when advected by the *horizontal* geostrophic flow. All the other dynamical variables may be obtained from potential vorticity as follows.

(i) Streamfunction, using (5.118b).

(ii) Velocity: $\boldsymbol{u} = \mathbf{k}\times\nabla\psi\ [\equiv \nabla^{\perp}\psi = -\nabla\times(\mathbf{k}\psi)]$.

(iii) Relative vorticity: $\zeta = \nabla^2\psi$.

(iv) Perturbation pressure: $\phi = f_0\psi$.

(v) Perturbation buoyancy: $b' = f_0\partial\psi/\partial z$.

The length scale $L_d = NH/f_0$, emerges naturally from the QG dynamics. It is the scale at which buoyancy and relative vorticity effects contribute equally to the potential vorticity, and is called the *deformation radius;* it is analogous to the quantity $\sqrt{gH}/f_0$ arising in shallow water theory. In the upper ocean, with $N \approx 10^{-2}\,\mathrm{s}^{-1}$, $H \approx 10^3\,\mathrm{m}$ and $f_0 \approx 10^{-4}\,\mathrm{s}^{-1}$, then $L_d \approx 100\,\mathrm{km}$. At high latitudes the ocean is much less stratified and f is somewhat larger, and the deformation radius may be as little as 30 km (see Fig. 9.11 on page 395, where the deformation radius is defined slightly differently). In the atmosphere, with $N \approx 10^{-2}\,\mathrm{s}^{-1}$, $H \approx 10^4\,\mathrm{m}$, then $L_d \approx 1000\,\mathrm{km}$. It is this order of magnitude difference in the deformation scales that accounts for a great deal of the quantitative difference in the dynamics of the ocean and the atmosphere. If we take the limit $L_d \to \infty$ then the stratified quasi-geostrophic equations reduce to

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = \nabla^2\psi + f. \tag{5.119}$$

This is the two-dimensional vorticity equation, identical to (4.67). The high stratification of this limit has suppressed all vertical motion, and variations in the flow become confined to the horizontal plane. Finally, we note that it is typical in quasi-geostrophic applications to omit the prime on the buoyancy perturbations, and write $b = f_0 \partial\psi/\partial z$; however, we will keep the prime in this chapter.

5.4.3 Buoyancy advection at the surface

The solution of the elliptic equation in (5.118) requires vertical boundary conditions on ψ at the ground and at the top of the atmosphere, and these are given by use of the thermodynamic equation. For a flat, slippery, rigid surface the vertical velocity is zero so that the thermodynamic equation may be written as

$$\frac{\mathrm{D}b'}{\mathrm{D}t} = 0, \qquad b' = f_0 \frac{\partial\psi}{\partial z}. \tag{5.120}$$

We apply this at the ground and at the tropopause, treating the latter as a lid on the lower atmosphere. In the presence of friction and topography the vertical velocity is not zero, but is given by

$$w = r\nabla^2\psi + \boldsymbol{u}\cdot\nabla\eta_b \tag{5.121}$$

where the first term represents Ekman friction (with the constant r proportional to the thickness of the Ekman layer) and the second term represents topographic forcing. The boundary condition becomes

$$\frac{\partial}{\partial t}\left(f_0\frac{\partial\psi}{\partial z}\right) + \boldsymbol{u}\cdot\nabla\left(f_0\frac{\partial\psi}{\partial z} + N^2\eta_b\right) + N^2 r\nabla^2\psi = 0, \tag{5.122}$$

where all the fields are evaluated at $z = 0$ or $z = H$, the height of the lid. Thus, the quasi-geostrophic system is characterized by the horizontal advection of potential vorticity in the interior and the advection of buoyancy at the boundary. Instead of a lid at the top, then in a compressible fluid such as the atmosphere we may suppose that all disturbances tend to zero as $z \to \infty$.

* *A potential vorticity sheet at the boundary*

Rather than regarding buoyancy advection as providing the boundary condition, it is sometimes useful to think of there being a very thin sheet of potential vorticity just above the ground and another just below the lid, specifically with a vertical distribution proportional to $\delta(z-\epsilon)$ or $\delta(z-H+\epsilon)$, where ϵ is small. The boundary condition (5.120) or (5.122) can be replaced by this, along with the condition that there are no variations of buoyancy at the boundary and $\partial\psi/\partial z = 0$ at $z = 0$ and $z = H$.[8]

To see this, we first note that the differential of a step function is a delta function. Thus, a discontinuity in $\partial\psi/\partial z$ at a level $z = z_1$ is equivalent to a delta function in potential vorticity there:

$$q(z_1) = \left[\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right]_{z_1-}^{z_1+}\delta(z-z_1). \tag{5.123}$$

Now, suppose that the lower boundary condition, given by (5.120), has some arbitrary distribution of buoyancy on it. We can replace this condition by the simpler condition $\partial\psi/\partial z = 0$ at $z = 0$, provided we also add to our definition of potential vorticity a term given by (5.123) with $z_1 = \epsilon$. This term is then advected by the horizontal flow, as are the other contributions. A buoyancy source at the boundary must similarly be treated as a sheet of potential vorticity source in the interior. Any flow with buoyancy variations over a horizontal boundary is thus equivalent to a flow with uniform buoyancy at the boundary, but with a spike in potential vorticity adjacent to the boundary. This approach brings notational and conceptual advantages, in that now everything is expressed in terms of potential vorticity and its advection. However, in practice there may be less to be gained, because the boundary terms must still be included in any particular calculation that is to be performed.

5.4.4 Quasi-geostrophy in pressure coordinates

The derivation of the quasi-geostrophic system in pressure coordinates is very similar to that in height coordinates, with the main difference coming at the boundaries, and we give only the results. The starting point is the primitive equations in pressure coordinates, (2.153). In pressure coordinates, the quasi-geostrophic potential vorticity is found to be

$$q = f + \nabla^2\psi + \frac{\partial}{\partial p}\left(\frac{f_0^2}{S^2}\frac{\partial\psi}{\partial p}\right), \tag{5.124}$$

where $\psi = \Phi/f_0$ is the streamfunction and Φ the geopotential, and

$$S^2 \equiv -\frac{R}{p}\left(\frac{p}{p_R}\right)^\kappa \frac{\mathrm{d}\widetilde{\theta}}{\mathrm{d}p} = -\frac{1}{\rho\theta}\frac{\mathrm{d}\widetilde{\theta}}{\mathrm{d}p}, \tag{5.125}$$

where $\widetilde{\theta}$ is a reference profile and a function of pressure only. In log-pressure coordinates, with $Z = -H\ln p$, the potential vorticity may be written as

$$q = f + \nabla^2\psi + \frac{1}{\rho_*}\frac{\partial}{\partial Z}\left(\frac{\rho_* f_0^2}{N_Z^2}\frac{\partial\psi}{\partial Z}\right), \tag{5.126}$$

where

$$N_Z^2 = S^2\left(\frac{p}{H}\right)^2 = -\left(\frac{R}{H}\right)\left(\frac{p}{p_R}\right)^\kappa \frac{\mathrm{d}\widetilde{\theta}}{\mathrm{d}Z} \tag{5.127}$$

is the buoyancy frequency and $\rho_* = \exp(-z/H)$. Temperature and potential temperature are related to the streamfunction by

$$T = -\frac{f_0 p}{R}\frac{\partial \psi}{\partial p} = \frac{H f_0}{R}\frac{\partial \psi}{\partial Z}, \tag{5.128a}$$

$$\theta = -\left(\frac{p_R}{p}\right)^{\kappa}\left(\frac{f_0 p}{R}\right)\frac{\partial \psi}{\partial p} = \left(\frac{p_R}{p}\right)^{\kappa}\left(\frac{H f_0}{R}\right)\frac{\partial \psi}{\partial Z}. \tag{5.128b}$$

In pressure or log-pressure coordinates, potential vorticity is advected along isobaric surfaces, analogous to the horizontal advection in height coordinates.

The surface boundary condition again is derived from the thermodynamic equation. In log-pressure coordinates this is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial \psi}{\partial Z}\right) + \frac{N_Z^2}{f_0} W = 0, \tag{5.129}$$

where $W = \mathrm{D}Z/\mathrm{D}t$. This is not the real vertical velocity, w, but it is related to it by

$$w = \frac{f_0}{g}\frac{\partial \psi}{\partial t} + \frac{RT}{gH} W. \tag{5.130}$$

Thus, choosing $H = RT(0)/g$, we have, at $Z = 0$,

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial Z} - \frac{N_Z^2}{g}\psi\right) + \boldsymbol{u}\cdot\nabla\frac{\partial \psi}{\partial Z} = -\frac{N^2}{f_0} w, \tag{5.131}$$

where

$$w = \boldsymbol{u}\cdot\nabla\eta_b + r\nabla^2\psi. \tag{5.132}$$

This differs from the expression in height coordinates only by the second term in the local time derivative. In applications where accuracy is not the main issue the simpler boundary condition $\mathrm{D}(\partial_Z\psi)/\mathrm{D}t = 0$ is sometimes used.

5.4.5 The two-level quasi-geostrophic system

The quasi-geostrophic system has, in general, continuous variation in the vertical direction (and horizontal, of course). By finite-differencing the continuous equations we can obtain a *multi-level* model, and a crude but important special case of this is the *two-level* model, also known as the Phillips model.[9] To obtain the equations of motion one way to proceed is to take a crude finite difference of the continuous relation between potential vorticity and streamfunction given in (5.118b). In the Boussinesq case (or in pressure coordinates, with a slight reinterpretation of the meaning of the symbols) the continuous expression for potential vorticity is

$$q = \zeta + f + \frac{\partial}{\partial z}\left(\frac{f_0 b'}{N^2}\right), \tag{5.133}$$

where $b' = f_0 \partial\psi/\partial z$. In the case with a flat bottom and rigid lid at the top (and incorporating topography is an easy extension) the boundary condition of $w = 0$ is satisfied by $\mathrm{D}\partial_z\psi/\mathrm{D}t = 0$ at the top and bottom. An obvious finite-differencing of (5.133) in the vertical direction (see Fig. 5.3) then gives

Informal Derivation of Stratified QG Equations

We will use the Boussinesq equations, but similar derivations could be given using the anelastic equations or pressure coordinates. The first ingredient is the vertical component of the vorticity equation, (4.66); in the Boussinesq version (or the pressure coordinate or anelastic versions) there is no baroclinic term and we have

$$\frac{D_3}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right). \tag{QG.1}$$

We now apply the assumptions on page 203. The advection and the vorticity on the left-hand side are geostrophic, but we keep the horizontal divergence (which is small) on the right-hand side where it is multiplied by the big term f. Furthermore, because f is nearly constant we replace it with f_0 except where it is differentiated. The second term (tilting) on the right-hand side is smaller than the advection terms on the left-hand side by the ratio $[UW/(HL)]/[U^2/L^2] = [W/H]/[U/L] \ll 1$, because w is small ($\partial w/\partial z$ equals the divergence of the ageostrophic velocity). We therefore neglect it, and given all this (QG.1) becomes

$$\frac{D_g}{Dt}(\zeta_g + f) = -f_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = f_0\frac{\partial w}{\partial z}, \tag{QG.2}$$

where the second equality uses mass continuity and $D_g/Dt = \partial/\partial t + \boldsymbol{u}_g \cdot \nabla$.

The second ingredient is the three-dimensional thermodynamic equation,

$$\frac{D_3 b}{Dt} = 0. \tag{QG.3}$$

The stratification is assumed to be nearly constant, so we write $b = \widetilde{b}(z) + b'(x, y, z, t)$, where $\widetilde{b}$ is the basic state buoyancy. Furthermore, because w is small it only advects the basic state, and with $N^2 = \partial\widetilde{b}/\partial z$ (QG.3) becomes

$$\frac{D_g b'}{Dt} + wN^2 = 0. \tag{QG.4}$$

Hydrostatic and geostrophic wind balance enable us to write the geostrophic velocity, vorticity, and buoyancy in terms of streamfunction ψ [$= p/(f_0\rho_0)$]:

$$\boldsymbol{u}_g = \mathbf{k} \times \nabla\psi, \qquad \zeta_g = \nabla^2\psi, \qquad b' = f_0\partial\psi/\partial z. \tag{QG.5}$$

The quasi-geostrophic potential vorticity equation is obtained by eliminating w between (QG.2) and (QG.4), and this gives

$$\boxed{\frac{D_g q}{Dt} = 0, \qquad q = \zeta_g + f + \frac{\partial}{\partial z}\left(\frac{f_0 b'}{N^2}\right)}. \tag{QG.6}$$

This equation is the Boussinesq version of (5.118), and using (QG.5) it may be expressed entirely in terms of the streamfunction, with $D_g \cdot /Dt = \partial/\partial t + J(\psi, \cdot)$. The vertical boundary conditions, at $z = 0$ and $z = H$ say, are given by (QG.4) with $w = 0$, with straightforward generalizations if topography or friction are present.

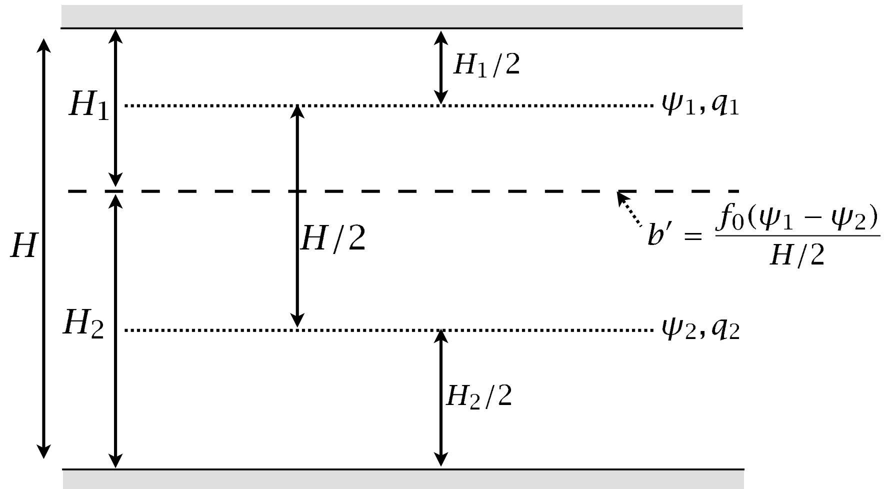

Fig. 5.3 The two-level quasi-geostrophic system with a flat bottom and rigid lid at which $w = 0$.

$$q_1 = \zeta_1 + f + \frac{2f_0^2}{N^2 H_1 H}(\psi_2 - \psi_1), \qquad q_2 = \zeta_2 + f + \frac{2f_0^2}{N^2 H_2 H}(\psi_1 - \psi_2). \tag{5.134}$$

In atmospheric problems it is common to choose $H_1 = H_2$, whereas in oceanic problems we might choose to have a thinner upper layer, representing the flow above the main thermocline. Note that the boundary conditions of $w = 0$ at the top and bottom are already taken care of in (5.134): *they are incorporated into the definition of the potential vorticity* — a finite-difference analogue of the delta-function construction of section 5.4.3. At each level the potential vorticity is advected by the streamfunction so that the evolution equation for each level is:

$$\frac{\mathrm{D}q_i}{\mathrm{D}t} = \frac{\partial q_i}{\partial t} + \boldsymbol{u}_i \cdot q_i = \frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \qquad i = 1, 2. \tag{5.135}$$

Models with more than two levels can be constructed by extending the finite-differencing procedure in a natural way.

Connection to the layered system

The two-level expressions, (5.134), have an obvious similarity to the *two-layer* expressions, (5.85). Noting that $N^2 = \partial\hat{b}/\partial z$ and that $b = -g\delta\rho/\rho_0$ it is natural to let

$$N^2 = -\frac{g}{\rho_0}\frac{\rho_1 - \rho_2}{H/2} = \frac{g'}{H/2}. \tag{5.136}$$

With this identification we find that (5.134) becomes

$$q_1 = \zeta_1 + f + \frac{f_0^2}{g' H_1}(\psi_2 - \psi_1), \qquad q_2 = \zeta_2 + f + \frac{f_0^2}{g' H_2}(\psi_1 - \psi_2). \tag{5.137}$$

These expressions are identical to (5.85) in the flat-bottomed, rigid lid case. Similarly, a multi-layered system with n layers is equivalent to a finite-difference representation with n levels. It should be said, though, that in the pantheon of quasi-geostrophic models the two-level and two-layer models hold distinguished places.

5.5 * QUASI-GEOSTROPHY AND ERTEL POTENTIAL VORTICITY

When using the shallow water equations, quasi-geostrophic theory could be naturally developed beginning with the expression for potential vorticity. Is such an approach possible for the stratified primitive equations? The answer is yes, although the algebra is more complicated, as we will see.

5.5.1 * Using height coordinates

Noting the general expression, (4.119), for potential vorticity in a hydrostatic fluid, the potential vorticity in the Boussinesq hydrostatic equations is given by

$$Q = \Big[(v_x - u_y)b_z - v_z b_x + u_z b_y + f b_z\Big], \tag{5.138}$$

where the x, y, z subscripts denote derivatives. Without approximation, we write the stratification as $b = \widetilde{b}(z) + b'(x, y, z, t)$, and (5.138) becomes

$$Q = [f_0 N^2] + [(\beta y + \zeta)N^2 + f_0 b'_z] + [(\beta y + \zeta)b'_z - (v_z b'_x - u_z b'_y)], \tag{5.139}$$

where, under quasi-geostrophic scaling, the terms in square brackets are in decreasing order of size. Neglecting the third term, and taking the velocity and buoyancy fields to be in geostrophic and thermal wind balance, we can write the potential vorticity as $Q \approx \widetilde{Q} + Q'$, where $\widetilde{Q} = f_0 N^2$ and

$$Q' = (\beta y + \zeta)N^2 + f_0 b'_z = (\beta y + \nabla^2\psi)N^2 + f_0^2 \frac{\partial^2 \psi}{\partial z^2}. \tag{5.140}$$

The potential vorticity evolution equation is then

$$\frac{\mathrm{D}Q'}{\mathrm{D}t} + w\frac{\partial \widetilde{Q}}{\partial z} = 0. \tag{5.141}$$

The vertical advection is important only in advecting the basic state potential vorticity $\widetilde{Q}$ and so, neglecting $w\partial Q'/\partial z$ and dividing by N^2, (5.141) becomes

$$\frac{\partial q_*}{\partial t} + \boldsymbol{u}_g \cdot \nabla q_* + \frac{w}{N^2}\frac{\partial \widetilde{Q}}{\partial z} = 0, \tag{5.142}$$

where $\hat{q}$ is

$$q_* = (\beta y + \zeta) + \frac{f_0}{N^2} b'_z. \tag{5.143}$$

This is the approximation to the (perturbation) Ertel potential vorticity in the quasi-geostrophic limit. However, it is not the same as the expression for the quasi-geostrophic potential vorticity, (5.118b) and, furthermore, (5.142) involves a vertical advection. (Thus, we might refer to the expression in (5.118) as the 'quasi-geostrophic pseudopotential vorticity', but the prefix 'quasi-geostrophic' alone normally suffices.) We can derive (5.118) by eliminating w between (5.142) and the quasi-geostrophic thermodynamic equation $\partial b'/\partial t + \boldsymbol{u}_g \cdot \nabla b' + w\partial\widetilde{b}/\partial z = 0$.

5.5.2 Using isentropic coordinates

An illuminating and somewhat simpler path from Ertel potential vorticity to the quasi-geostrophic equations goes by way of isentropic coordinates.[10] We begin with the isentropic expression for the Ertel potential vorticity of an ideal gas,

$$Q = \frac{f+\zeta}{\sigma}, \tag{5.144}$$

where $\sigma = -\partial p/\partial\theta$ is the thickness density (which we will just call the thickness), and in adiabatic flow the potential vorticity is advected along isopycnals. We now employ quasi-geostrophic scaling to derive an approximate equation set from this. First, assume that variations in thickness are small compared with the reference state, so that

$$\sigma = \widetilde{\sigma}(\theta) + \sigma', \qquad |\sigma'| \ll |\sigma|, \tag{5.145}$$

and similarly for pressure and density. Assuming also that the variations in the Coriolis parameter are small, then on the β–plane (5.144) becomes

$$Q \approx \left[\frac{f_0}{\widetilde{\sigma}}\right] + \left[\frac{1}{\widetilde{\sigma}}(\zeta + \beta y) - \frac{f_0}{\widetilde{\sigma}}\frac{\sigma'}{\widetilde{\sigma}}\right]. \tag{5.146}$$

We now use geostrophic and hydrostatic balance to express the terms on the right-hand side in terms of a single variable, noting that the first term does not vary along isentropic surfaces. Hydrostatic balance is

$$\frac{\partial M}{\partial \theta} = \Pi, \tag{5.147}$$

where $M = c_pT + gz$ and $\Pi = c_p(p/p_R)^\kappa$. Writing $M = \widetilde{M}(\theta) + M'$ and $\Pi = \widetilde{\Pi}(\theta) + \Pi'$, where $\widetilde{M}$ and $\widetilde{\Pi}$ are hydrostatically balanced reference profiles, we obtain

$$\frac{\partial M'}{\partial\theta} = \Pi' \approx \frac{\mathrm{d}\widetilde{\Pi}}{\mathrm{d}p}p' = \frac{1}{\theta\widetilde{\rho}}p', \tag{5.148}$$

where the last equality follows using the equation of state for an ideal gas and $\widetilde{\rho}$ is a reference profile. The perturbation thickness field may then be written as

$$\sigma' = -\frac{\partial}{\partial\theta}\left(\widetilde{\rho}\theta\frac{\partial M'}{\partial\theta}\right). \tag{5.149}$$

Geostrophic balance is $\boldsymbol{f}_0 \times \boldsymbol{u} = -\nabla_\theta M'$ where the velocity, and the horizontal derivatives, are along isentropic surfaces. This enables us to define a flow streamfunction by

$$\psi \equiv \frac{M'}{f_0} \tag{5.150}$$

and we can then write all the variables in terms of ψ:

$$\begin{gathered} u = -\left(\frac{\partial\psi}{\partial y}\right)_\theta, \qquad v = \left(\frac{\partial\psi}{\partial x}\right)_\theta, \\ \zeta = \nabla_\theta^2\psi, \qquad \sigma' = -f_0\frac{\partial}{\partial\theta}\left(\widetilde{\rho}\theta\frac{\partial\psi'}{\partial\theta}\right). \end{gathered} \tag{5.151}$$

Using (5.146) (5.150) and (5.151), the quasi-geostrophic system in isentropic coordinates may be written

$$\boxed{\begin{aligned} \frac{\mathrm{D}q}{\mathrm{D}t} &= 0 \\ q = f + \nabla_\theta^2\psi &+ \frac{f_0^2}{\widetilde{\sigma}}\frac{\partial}{\partial\theta}\left(\widetilde{\rho}\theta\frac{\partial\psi}{\partial\theta}\right) \end{aligned}}, \tag{5.152a,b}$$

where the advection of potential vorticity is by the geostrophically balanced flow, along isentropes. The variable q is an approximation to the second term in square brackets in (5.146), multiplied by $\widetilde{\sigma}$.

Projection back to physical-space coordinates

We can recover the height or pressure coordinate quasi-geostrophic systems by projecting (5.152) on to the appropriate coordinate. This is straightforward because, by assumption, the isentropes in a quasi-geostrophic system are nearly flat. Recall that [cf. (2.142)] a transformation between vertical coordinates may be effected by

$$\left.\frac{\partial}{\partial x}\right|_\theta = \left.\frac{\partial}{\partial x}\right|_p + \left.\frac{\partial p}{\partial x}\right|_\theta \frac{\partial}{\partial p}, \tag{5.153}$$

but the second term is $\mathcal{O}(Ro)$ smaller than the first one because, under quasi-geostrophic scaling, isentropic slopes are small. Thus $\nabla_\theta^2\psi$ in (5.152b) may be replaced by $\nabla_p^2\psi$ or $\nabla_z^2\psi$. The vortex stretching term in (5.152) becomes, in pressure coordinates,

$$\frac{f_0^2}{\widetilde{\sigma}}\frac{\partial}{\partial\theta}\left(\widetilde{\rho}\theta\frac{\partial\psi}{\partial\theta}\right) \approx \frac{f_0^2}{\widetilde{\sigma}}\frac{\mathrm{d}\widetilde{p}}{\mathrm{d}\theta}\frac{\partial}{\partial p}\left(\widetilde{\rho}\theta\frac{\mathrm{d}\widetilde{p}}{\mathrm{d}\theta}\frac{\partial\psi}{\partial p}\right) = \frac{\partial}{\partial p}\left(\frac{f_0^2}{S^2}\frac{\partial\psi}{\partial p}\right) \tag{5.154}$$

where S^2 is given by (5.125). The expression for the quasi-geostrophic potential vorticity in isentropic coordinates is thus approximately equal to the quasi-geostrophic potential vorticity in pressure coordinates. This near-equality holds because the isentropic expression, (5.152b), does not contain a component proportional to the mean stratification: the second square-bracketed term on the right-hand side (5.146) is the only dynamically relevant one, and its evolution along isentropes is mirrored by the evolution along isobaric surfaces of quasi-geostrophic potential vorticity in pressure coordinates.

5.6 * ENERGETICS OF QUASI-GEOSTROPHY

If the quasi-geostrophic set of equations is to represent a real fluid system in a physically meaningful way, then it should have a consistent set of energetics. In particular, the total energy should be conserved, and there should be analogs of kinetic and potential energy and conversion between the two. We now show that such energetic properties do hold, using the Boussinesq set as an example.

Let us write the governing equations as a potential vorticity equation in the interior,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right)\right] + \beta\frac{\partial\psi}{\partial x} = 0, \qquad 0 < z < 1, \tag{5.155}$$