Theory Seminar: Garrett (1976)

**Generation of Langmuir circulations by surface waves—
a feedback mechanism**

by Christopher Garrett[1]

http://images.hanneketravels.net/golven.jpg

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G76: A wave-current interaction model

Similar to CL 1976: irrotational waves interact with a weak rotational current

But! Theory includes effects of mean flow on waves in the form of wave refraction

Theory largely focuses on properties of linear surface gravity waves

Through a (parameterized) feedback process, these interactions are capable of producing Langmuir circulations.

Outline

- The effects of currents on waves
- The effects of waves on mean flow
- Application to Langmuir Circulations
- Instability
- Limitations, inconsistencies, and relationship to CL 76

Geometrical Optics

General idea: When waves encounter a current, their speed, direction, wavelength, height and shape can be altered by the interaction. For linear/weakly nonlinear waves, many of these features can be described by *geometrical optics.*

Consider a wave that does not vary *very much* in a local region of space and time (WKB approximation).

If this is true, we introduce ***rays***, which are lines such that the direction of propagation is tangent to them at any point.

We can say that water waves (or wave groups) are propagated along the rays, and ignore their exact wave nature.

Geometrical Optics

Consider a function $\Theta(x,y,t)$, from which we have the relationship (equality of mixed partials):

$$\boldsymbol{\nabla}\left(\frac{\partial\theta}{\partial t}\right) - \frac{\partial}{\partial t}(\boldsymbol{\nabla}\theta) = \mathbf{0} \qquad \text{where} \qquad \boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

Next, we define the angular frequency ω and the wavenumber $\boldsymbol{k}=(k,l)$ as

$$\omega = -\frac{\partial\theta}{\partial t}; \qquad \mathbf{k} = \boldsymbol{\nabla}\theta$$

If $\omega=\omega(\mathrm{k},l)$, then

$$\frac{d\mathbf{x}}{dt} = \frac{\partial\omega}{\partial\mathbf{k}}$$

Group velocity

$$\frac{\partial\mathbf{k}}{\partial t} + \boldsymbol{\nabla}\omega = 0$$

Conservation of waves

Geometrical Optics

So far all of this is general, as we have just assumed the existence of a phase $\Theta = \Theta(x,y,t)$ and that the wave properties do not change very much over an oscillation.

The *physics* of the system enters through the dispersion relation, that is, how ω relates to $\boldsymbol{k}$ (or other wave properties like wave slope), and possibly ($\mathbf{x}$,t), i.e. the medium through which the waves propagate.

For surface gravity waves on a weak current, we have

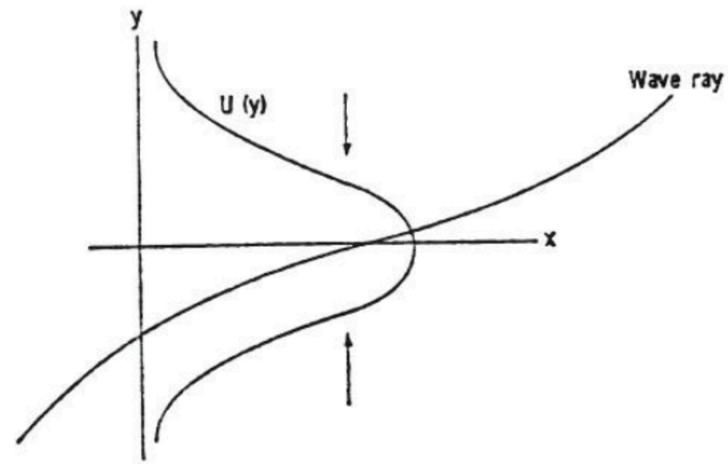
$$\omega = \omega' + \mathbf{U} \cdot \mathbf{k}$$

$$\omega' = \sqrt{g}(k^2 + l^2)^{\frac{1}{4}} \qquad \mathbf{U} = (U(y), V(y))$$

and $\mathbf{k} = (k, l)$.

Geometrical Optics

With these assumptions, Garrett writes down the relevant ray equations as:



$$\frac{\partial \omega}{\partial t} + \mathbf{c_g} \cdot \boldsymbol{\nabla} \omega = 0$$

$$\frac{\partial k}{\partial t} + \mathbf{c_g} \cdot \boldsymbol{\nabla} k = 0$$

$$\frac{\partial l}{\partial t} + \mathbf{c_g} \cdot \boldsymbol{\nabla} l = \underbrace{-k \frac{dU}{dy}}_{\text{rotation}} \underbrace{- l \frac{dV}{dy}}_{\text{stretching}}$$

where

$$\mathbf{c_g} = \mathbf{U} + \left(\frac{\partial \omega'}{\partial k}, \frac{\partial \omega'}{\partial l} \right)$$

Geometrical Optics

l can be found directly from the dispersion relationship, i.e.

$$\omega = \sqrt{g}(k^2 + l^2)^{\frac{1}{4}} + Uk + Vl$$

Defining the angle between l,k as θ, i.e. $\theta = \tan^{-1}\left(\frac{l}{k}\right)$

We find $\omega = \sqrt{g|k|}\sqrt{|\sec\theta|} + k(U + V\tan\theta)$

In the limit where $|\mathbf{U}|/c<<1$, then to first order in this small parameter

$$\theta = \theta_0 - 2(\cos\theta_0)^{\frac{1}{2}}(\hat{U}\cot\theta_0 + \hat{V})$$

Which, for θ_o approaching 0, diverges

Wave Action Conservation

Bretherton and Garrett (1967): ***Wave action***

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega'}\right) + \boldsymbol{\nabla}\cdot\left(\mathbf{c_g}\frac{E}{\omega'}\right) = 0$$

$$\Longleftrightarrow \quad \frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot(\mathbf{c_g}E) + \frac{1}{2}S_{ij}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) = 0$$

Radiation stress S_{ij}: **Excess flux of momentum due to waves** (gradients of which provide a force)

In deep water, for linear monochromatic waves, this reduces to

$$S_{ij} = \frac{E k_i (c_g')_j}{\omega'}$$

where $E = \frac{1}{2}\rho g a^2$

Wave Action Conservation

For weak currents, the wave action equation reduces to

$$E/E_0 = 1 + 4(\cos\theta_0)^{\frac{1}{2}}\,(\hat{U}\cot\theta_0\cot 2\theta_0 - \hat{V}\tan\theta_0).$$

which for θ_o small is dominated by the U term.

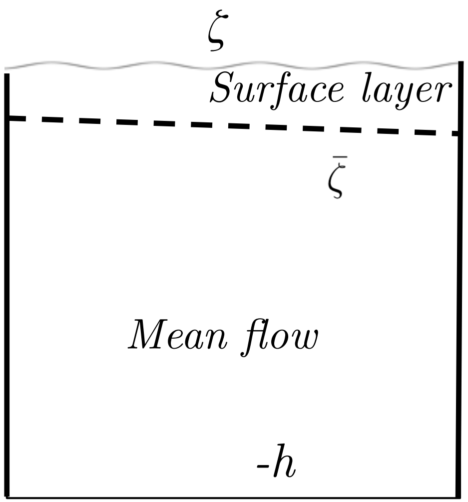
This implies the wave energy (and hence wave amplitude) is largest at maximum U.

But similar to the ray equations, this breaks down as θ_o goes to 0.

Effects of waves on the mean flow

The horizontal momentum equations are (α,β = x,y; 3=z)

$$\frac{\partial}{\partial t}(\rho u_\alpha) + \frac{\partial}{\partial x_\beta}(\rho u_\alpha + p\delta_{\alpha\beta}) + \frac{\partial}{\partial z}(\rho u_\alpha u_3) = 0$$



Decompose the flow into mean (phase averaged) and fluctuating components

$$u_\alpha = U_\alpha + u'_\alpha$$

Assume U is depth independent over a few times the e-folding scale of the waves

Mean flow equations

Integrating this equation over depth, and phase averaging over the waves, we find

Now the mean flow is time dependent!

$$\frac{\partial \bar{M}_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta} \int_{-h}^{\zeta} (\rho U_\alpha U_\beta + p^m \delta_{\alpha\beta})\, dz = -\frac{\partial}{\partial x_\beta}(U_\alpha M_\beta + U_\beta M_\alpha + S_{\alpha\beta})$$

where

$$\overline{M}_\alpha = \overline{\int_{-h}^{\zeta} \rho u_\alpha \, dz} = M_\alpha^m + M_\alpha$$

$$\underbrace{M_\alpha^m = \int_{-h}^{\zeta} \rho U_\alpha \, dz;}_{\text{Momentum in currents}} \qquad \underbrace{M_\alpha = \overline{\int_{\bar{\zeta}}^{\zeta} \rho u'_\alpha \, dz}}_{\text{Wave momentum}} = \frac{E k_\alpha}{\omega'}$$

M_α is equivalent to the vertical integral of the Stokes drift times the density

Mean flow equations

We want an evolution equation for $M_a{}^m$, so that we must remove the change in the waves momentum:

$$\frac{\partial M_\alpha}{\partial t} = -\frac{\partial}{\partial x_\beta}\left(M_\alpha(U_\beta + (c_g')_\beta\right) - M_\alpha \frac{\partial U_\beta}{\partial x_\alpha}$$

So that the governing equation for the mean flow is

$$\frac{\partial M_\alpha^m}{\partial t} + \frac{\partial}{\partial x_\beta}\int_{-h}^{\zeta}\left(\rho U_\alpha U_\beta + p^m \delta_{\alpha\beta}\right)\, dz = F_\alpha^m$$

$$\mathbf{F}^m = -\underbrace{\mathbf{U}\boldsymbol{\nabla}\cdot\mathbf{M}}_{\text{Divergence of mass flux term}} + \underbrace{\mathbf{M}\times(\boldsymbol{\nabla}\times\mathbf{U})}_{\text{Vertically integrated vortex force}}$$

Mean flow equations

We also have the kinematic condition and dynamic boundary conditions for the mean flow

$$\frac{\partial \bar{\zeta}}{\partial t} + U_\alpha \frac{\partial \bar{\zeta}}{\partial x_\alpha} - U_3 = \underbrace{-\frac{1}{\rho}\frac{\partial M_\alpha}{\partial x_\alpha}} \qquad @\ z = \bar{\zeta}$$

Upwelling term due to mass flux

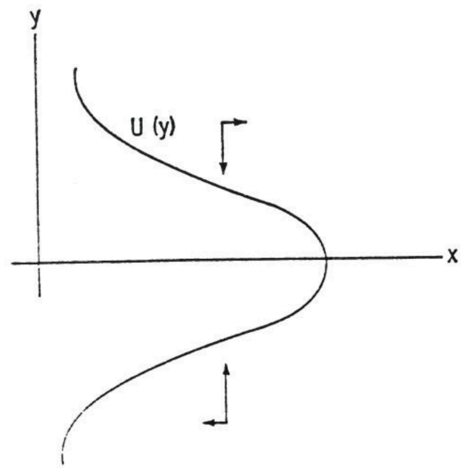
$$p^m = \overline{p}_a \qquad @z = \bar{\zeta}$$

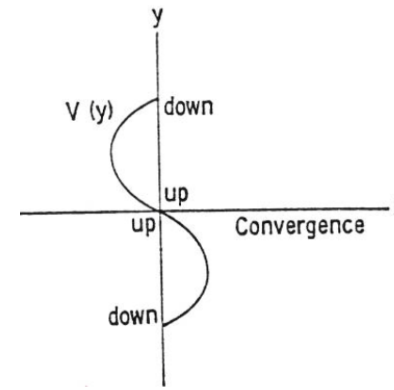
Application to Langmuir Circulations

a) For weak currents, and momentum independent of x, $\mathbf{F^m}$ reduces to

$$\mathbf{F}^m = [-E_0 \tan\theta_0 (\cos\theta_0)^{1/2}, E_0 (\cos\theta_0)^{1/2}] d\hat{U}/dy.$$

That is it leads to convergence in the region of the current maximum





b) Mass flux divergence term in the kinematic BC:

$$dM_y/dy = -E_0(k/g)^{1/2} [3 \sin\theta_0 \, d\hat{U}/dy + \sec\theta_0 (2+3\sin^2\theta_0) d\hat{V}/dy]$$

This opposes the circulation.

Typical magnitudes of the source terms

The y-directional component of $\mathbf{F}^{m}$ is

$$F_y{}^m = E_0 \, (\cos\theta_0)^{\frac{1}{2}} \, (k/g)^{\frac{1}{2}} dU/dy \approx \tfrac{1}{2}\,\rho g a_0{}^2 (U_0/c_0)\, L^{-1}$$

Consider the scales:

$a_o = 0.25\text{m}$, $c_o = 5\text{ms}^{-1}$, $L = 3\text{m}$, $U_o = 0.01\text{ms}^{-1}$

so that $\mathbf{F_y{}^m = 0.2Nm^{-2}}$

For a wind stress of 5ms^{-1}, the force is $\mathbf{0.03Nm^{-2}}$.

The magnitude of the upwelling for small θ_o is $V_o = 0.01\text{ms}^{-1}$ implies upwelling velocity is $8\times10^{-5}\ \text{ms}^{-1}$, which is *negligible* compared to observed downwelling speeds of around $0.05\ \text{ms}^{-1}$.

Wave dissipation

The vortex force leads to a convergence of currents at the maximum of U, but for there to be LC, U must be reinforced.

Wave dissipation can provide this reinforcement, as the momentum lost from the wave field will go into the currents, and is greatest where wave energy is greatest, hence where U is greatest.

Assume energy loss rate is γE, with corresponding momentum loss γM_α, which takes momentum from the wave field and puts it into the mean flow, ie we add it to $\mathbf{F}^m$.

Wave dissipation

For weak currents, and using the relation E=Mc,

$$M_x = E_0(k/g)^{\frac{1}{2}}(\cos\theta_0)^{\frac{1}{2}}[1+2\hat{U}\cot^2\theta_0(\cos\theta_0)^{\frac{1}{2}}(1-\tfrac{1}{2}\tan^2\theta_0)$$
$$-3\hat{V}\sin\theta_0(\cos\theta_0)^{-\frac{1}{2}}]$$

Which is dominated by the U term for θ_o small, i.e. the momentum input due to breaking is proportional to U.

Therefore, for a narrow directional spectrum about **x**, the dominant effects of waves on mean flow are

i) Force in y-direction proportional to dU/dy

ii) Force in x-direction, proportional to U

For wavelengths short compared to the current, these act like surface stresses on the mean flow

The instability

Garrett considers a system subject to the surface stress:

$$\tau(y) = (aU, bdU/dy) \text{ at } z = 0.$$

Governing equations

Linearized NS $\quad \partial\mathbf{U}/\partial t + (1/\rho)\nabla P = \nu\nabla^2\mathbf{U},\ \nabla.\mathbf{U} = 0$

BC $\quad \mu\partial U/\partial z = aU,\ \mu\partial V/\partial z = b\partial U/\partial y,\ W = 0 \text{ at } z = 0$

and $\mathbf{U} \to 0$ as $z \to -\infty$.

Garrett looks for solutions of the form

$$\mathbf{U}(y,z,t) = Re\ \mathbf{u}(z)e^{i\beta y}e^{qt},\ P(y,z,t) = Re\ p(z)e^{i\beta y}e^{qt}$$

The instability

Note, the governing equations in x and (y,z) are decoupled (c.f. Craik 1977).

The problem for U has solution:

$$\mathbf{U}(y,z,t) = Re\ \mathbf{u}(z)e^{i\beta y}e^{qt},$$

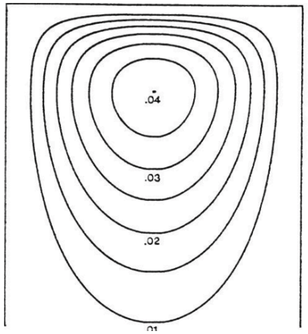
$$u = u_0 \exp[(a/\mu)z],\ q = \nu[(a/\mu)^2 - \beta^2]$$

The most unstable mode occurs when β=0.

In the transverse direction, the stream function has solution:

$$\Psi(y,z,t) = \mathrm{Re}\ \psi(z)e^{i\beta y}e^{qt}$$

$$\psi(z) = i\beta b(q\rho)^{-1}u_0\{\exp[(\beta^2+q/\nu)^{\frac{1}{2}}z] - \exp(\beta z)\}$$



Model limitations

Model ignores internal reflections (problem was considered by J. Smith 1983, McKee 1974, Peregrine 1976)

Inconsistencies in governing equations for $\mathbf{U}$ (time dependence, viscosity, etc...)

Parameterization of momentum loss due to breaking

Accuracy of Garrett's predictions

Veron and Melville (2001)

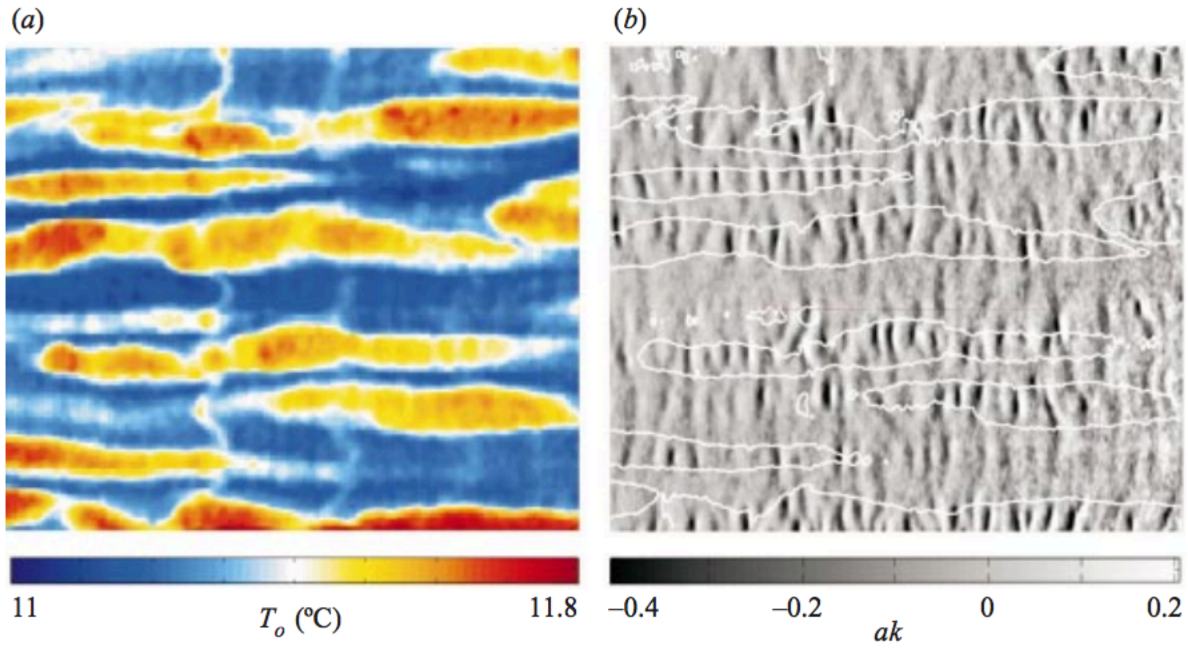
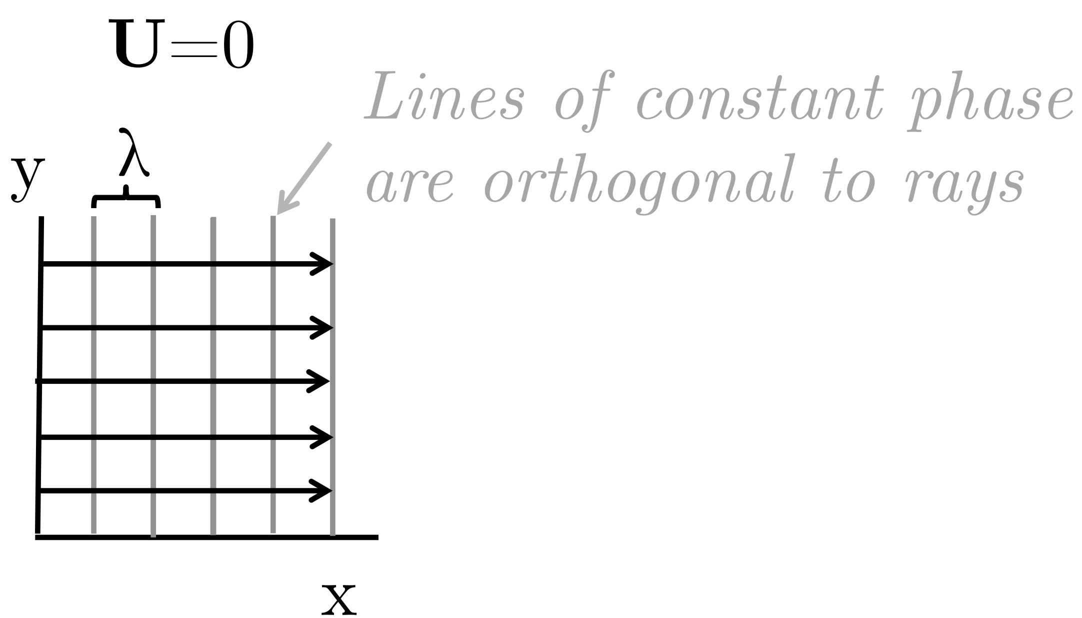


FIGURE 15. (*a*) Surface temperature (from figure 4*c*), and (*b*) along-wind slope at the same location at time $t = 19.8$ s. Note the correlation between the warm upwelling regions and the regions of larger wave slope. Wind and waves are travelling left to right. The image size is 36.8 cm × 27 cm.

Geometrical Optics

Level sets of Θ correspond to waves (e.g. we can associate these with crests or troughs, provided they do not disappear).

Example: Monochromatic linear deep-water surface gravity waves:



Wave Action Conservation

When waves interact with a current, they can exchange momentum and energy, hence energy is not necessarily conserved

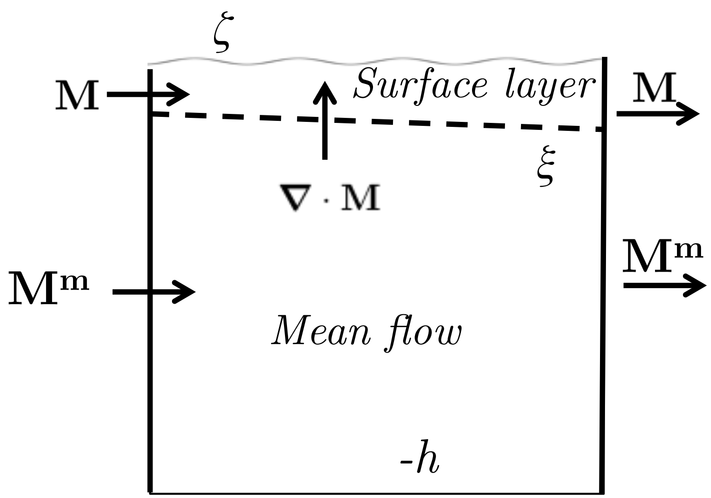
However, if the interaction is weak, there is still a conserved quantity, namely **wave action**

Simplest example (Einstein 1911) is a pendulum with varying string length

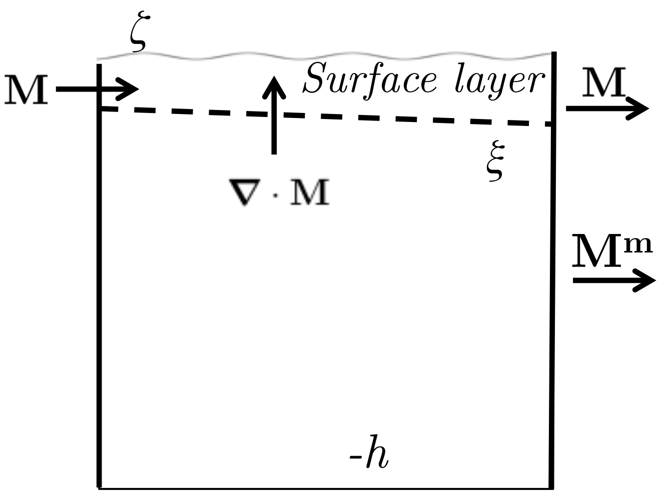
Mean flow equations

Next we consider the effects of the *waves on the mean flow*

Mass Balance



Momentum Balance



Geometrical Optics

Garrett: Section II: We consider linear deep-water waves, with wavelength λ, on a slowly varying (time independent) background current **U**=**U**(δy), δ=O(1/L)<<1/λ

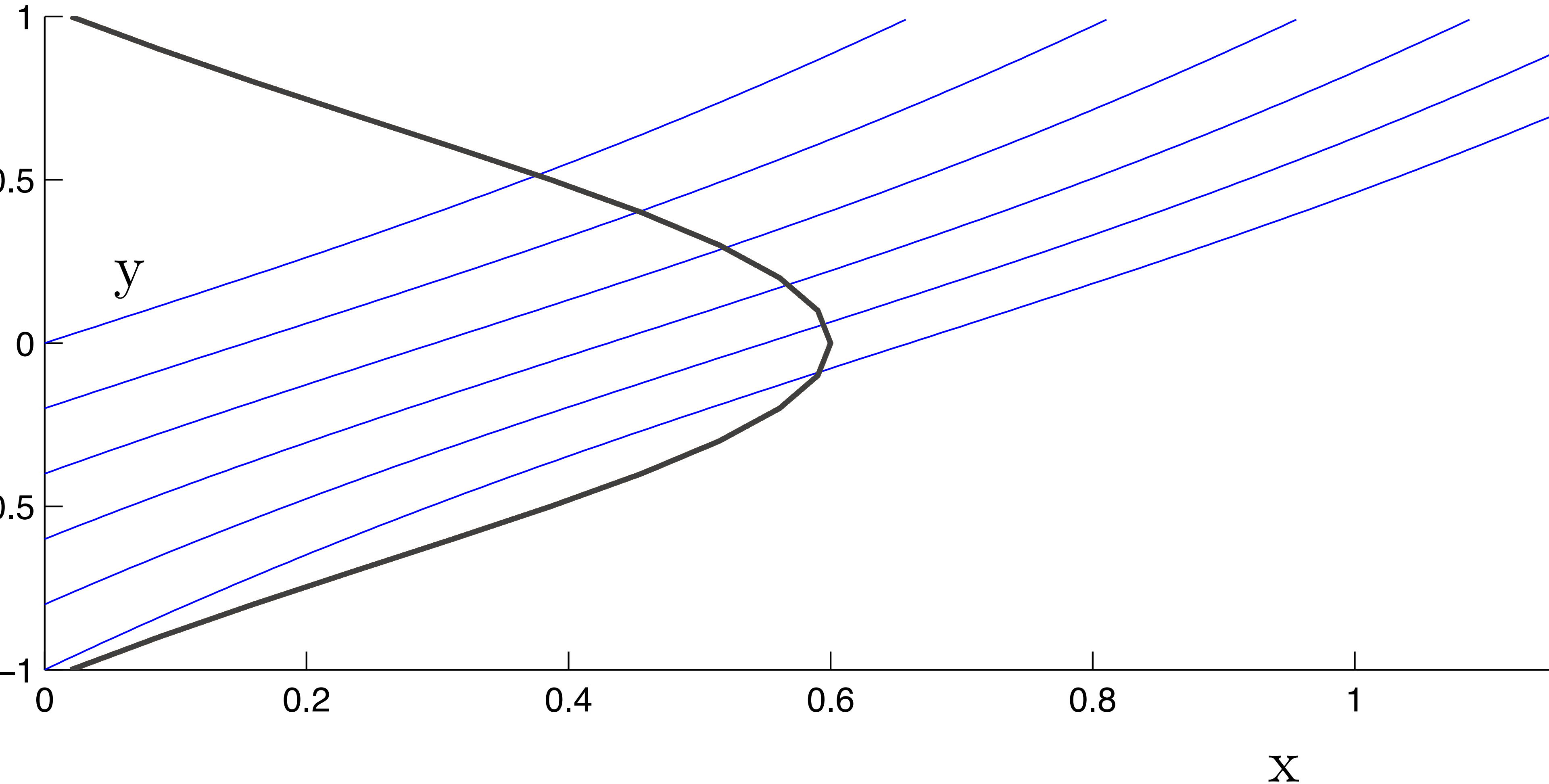
From the (irrotational) governing equations, one finds

$$\omega = \omega' + \mathbf{U} \cdot \mathbf{k}$$

where $\omega' = \sqrt{g}(k^2 + l^2)^{\frac{1}{4}}$ $\mathbf{U} = (U(y), V(y))$

and **k**=(k,l)

Geometrical Optics



Mirror symmetric about y =0

Governing equations for surface gravity waves

We *ignore* the effects of capillarity, viscosity, assume a constant density, and planetary rotation.
Let pressure be p^a at the interface $z=\zeta$, and tangential stress be 0

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = \mathbf{u} \cdot \hat{z}; \qquad @\ z = \zeta$$

$$p = p^a; \qquad @\ z = \zeta$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - g\hat{z}; \qquad \nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u} \cdot \hat{z} = 0; \qquad @z = -H$$

Governing equations for surface gravity waves

In Garrett's work, we will often consider results from irrotational wave theory

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \zeta = \mathbf{u} \cdot \hat{z}; \qquad @\ z = \zeta$$

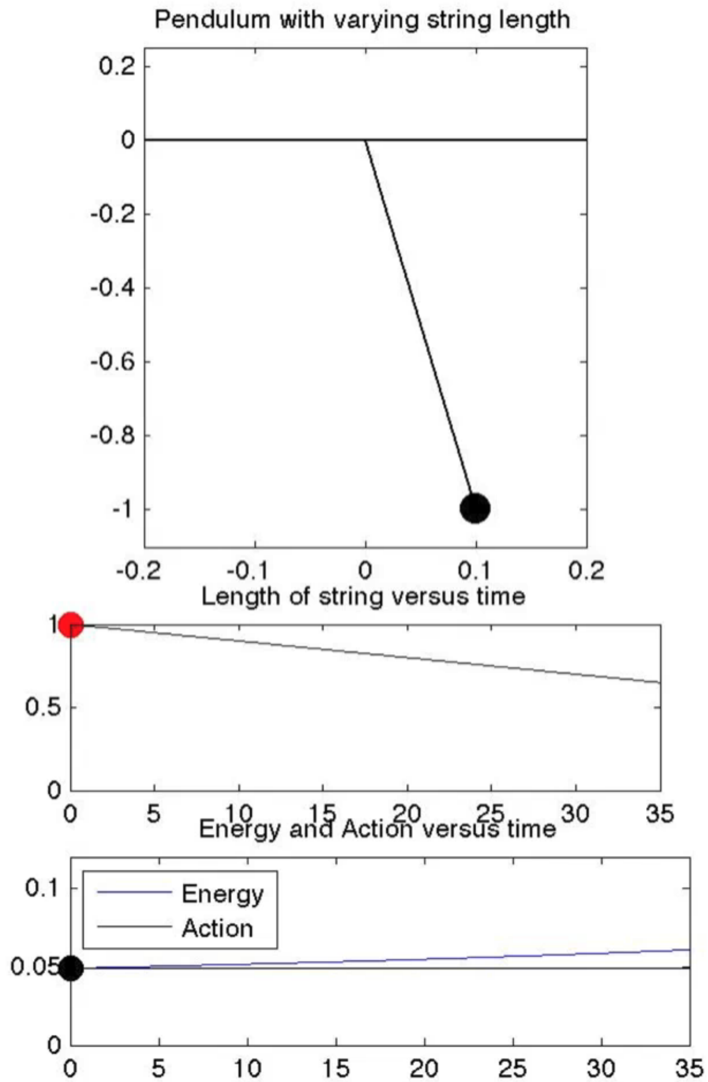
$$-\frac{p^a}{\rho} = \phi_t + \frac{1}{2}(\boldsymbol{\nabla} \phi)^2 + gz; \qquad @z = \zeta$$

$$\boldsymbol{\nabla}^2 \phi = 0$$

$$\mathbf{u} \cdot \hat{z} = 0; \qquad @z = -H$$

Wave Action Conservation

Simple example: Pendulum with varying string length



Wave Action Conservation

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