Shorter Contribution

Hydrodynamical Analogy to $E = mc^2$

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For many years the study from certain viewpoints of the theory of gravity waves in a simple fluid has formed one of my scientific occupations. Of late I have been stimulated in this direction by the work of others, for example through the accomplishments of my friend, Professor R. LONG, who in 1956 succeeded in the specification of the more exact solution for the solitary wave problem. Also, since some of my former results have been much misunderstood, it seems quite in order at this time to present a brief historical review of the general status of the subject leading up to this material, and then to point to certain less well known connotations of the newer findings.

If one concentrates attention upon *exact* theoretical solutions of problems in this type of wave motion, there are few indeed that are known. The example of plane periodic irrotational surface waves stands almost alone. How was success attained in this instance? The problem is an old one. Its development is closely connected with the growth of hydrodynamics itself. Mathematically it consists of obtaining a solution of the Laplace equation fulfilling a nonlinear boundary condition prescribed by the uniformity of the pressure at the free surface. Although it may be rather simply stated, historically its elucidation required the better part of a century. As is often the case in theoretical hydrodynamics, there were no essentially radical changes in the basic approach during this time. Rather, there was a gradual erosion of difficulties, so that progressively newer results constituted a rounding out and amplification of previous knowledge.

One may note first the approximate solution given by AIRY (1845). This was obtained by linearization of the problem, a technique so often resorted to by physicists and mathematicians. STOKES (1847) proposed a systematic method for computing higher approximations beyond the linear one, although he did not establish the convergence of his successive approximations. The problem was then taken up by numerous investigators, notably by RAYLEIGH (e.g., 1911), in order to vindicate or disprove the results of Stokes. Although valuable information was gained during this period, the main point at issue remained unresolved. Final success was reserved for the efforts of the renowned Italian mathematician T. LEVI-CIVITA, who in 1925 furnished the necessary proof of convergence of the Stokes series for the case of a deep fluid. Shortly thereafter STRUIK (1926) furnished the proof for a fluid of arbitrary depth. A landmark was thus erected.

The subject is, however, far from exhausted for further theoretical inquiry. Not only are there efforts being made to simplify and improve the methods of calculation of specific examples (see, e.g., DAVIES 1951), but also the physical significance of the results constitutes a branch which has barely been entered upon. It is on this latter topic that the interest of the writer has been focussed. One may likewise observe that a related problem, namely that of the solitary wave, has received the beginnings of a comparable exact treatment only recently, as already stated, by Long (and others), who, incidentally, verified certain inequalities previously derived in my own studies. Since the subject of the solitary wave is somewhat specialized, no further discussion of my study of it appears herein; the reader who is interested is referred to STARR (1947b).

It was noted already by Stokes that periodic waves in a deep fluid are associated with a mass transport in the direction of motion of the waves. This physical property may also be described by saying that the waves have a certain momentum M per wavelength. The qualitative identity of these two statements is apparent intuitively. Not so obvious however is the fact that the factor of proportionality involved between these two quantities is the wave speed c, albeit that a simple kinematic analysis is sufficient for its establishment (STARR 1948). We have thus that, if m is the mass of fluid transported across a fixed vertical during a wave period

$$M = cm \qquad (1)$$

A relationship noted by Levi-Civita, which moreover may easily be verified directly from the differential equations of the problem (STARR 1947a, 1947b) connects the *kinetic energy* e of the motions per wavelength (and, as in the other quantities, per unit length along the crests) with the momentum and the wave speed. The assertion states that

$$M\,c = 2e \qquad (2)$$

The kinetic energy denoted here by e may be conceived of as being due in part to the contribution of the horizontal motions of particles, namely e_x, and in part to the contribution of the vertical motions e_z, so that $e \equiv e_x + e_z$. Through simple manipulations of the equations of motion and continuity it may be shown that, without approximation, the *potential energy* v per wavelength is related as follows to these quantities (STARR 1947b):

$$e_x - e_z = 2\ (e - v) \equiv 2\varepsilon \qquad (3)$$

We find that, in agreement with intuitive estimates made otherwise, the horizontal kinetic energy e_x is greater than the vertical e_z, since v is never larger than e (RAYLEIGH, 1911; PLATZMAN, 1947). When the special case of small wave amplitude is approached, e_z approaches equality with e_x (the particles then move more nearly uniformly in circles), leading to the well known fact that the linearized theory yields an equality of kinetic and potential energies. According to Platzman, in the case of extreme wave height the difference ε in (3) is about twelve per cent of v. The power series for ε starts with a term of the fourth order.

Much has been written concerning the transmission of energy in the direction of propagation of surface waves of the kind here visualized. Most of such discussions deal with the subject only from the standpoint of linear theory, or else through the application of methods containing comparable systematic approximations. If the subject is approached at the beginning from the point of view of the exact physical processes which take place, it can be seen that the total energy flow receives contributions from (a) an advection $\{e\}$ across a given vertical during one wave period of fluid with kinetic energy, (b) a similar advection $\{v\}$ of fluid with potential energy, and (c) the work $\{p\}$ done across the given vertical by pressure forces in virtue of the components of horizontal velocity (see STARR and PLATZMAN, 1948). Several important relations may be derived which pertain to these three actions, again from the unmutilated differential equations. Since the particular formulae are not to be used directly here for the elucidation of the subsequent discussion, only a brief statement of them is quoted.

The transport of kinetic energy is given by

$$\{e\} = e_x - e_z \qquad (4)$$

or, in virtue of (3), $\{e\} = 2\varepsilon$. The transport of potential energy turns out to be

$$\{v\} = e_x - e_z \qquad (5)$$

so that it is the same as $\{e\}$. The work done by pressure forces is

$$\{p\} = 2e_z \qquad (6)$$

The total energy transport thus turns out to be $2e_x$. Owing to the situation that for small amplitudes e_z approaches e_x and v approaches e, one regains the classic notion that one half the total energy per wavelength is transported during a wave period. Actually because $e_x \geqq e_z$ and $e \geqq v$, the transport is larger than one half in the case of finite disturbances.

With these remarks the aforementioned recapitulation is ended. It is however still my desire to take here an opportunity to dispel an unfortunate misapprehension which has gained some ground—namely that the material presented is merely the result of *arbitrary assumptions* made in the several derivations. This attitude has patently no foundation in fact. The methods used are familiar ones, of which there are many other illustrations in hydrodynamics, for example such as the Kutta-Joukowski theory for the lift of an airfoil (see, e.g., LAMB 1932).

The relation (2) is reminiscent of the comparable phenomenon in the theory of electrodynamics, namely that of light pressure, predicted by Maxwell and detected experimentally later. When the momentum M is eliminated between (1) and (2) the result is gained that

$$e = {}^1/_2\, m\, c^2 \qquad (7)$$

a fact which superficially appears simple, until it is recalled that m is not the total mass of water in motion, and that c is not the particle velocity but rather the phase velocity. Apparently the complicated motions of large amounts of fluid which give rise to the value of the kinetic energy e are equivalent to the motion of simple translation of a mass equal to that transmitted in a wave period, moving with the wave speed.

We now revert to (3) and the remarks there made in regard to it, i.e., that $v \leqq e$. Since the total energy E of the waves per wavelength is the sum of the potential and kinetic, which energy forms the particles of fluid change periodically, the inference now follows from (7) that

$$\boxed{E \leqq m\, c^2} \qquad (8)$$

This statement is of essentially the same form as the epoch-making generalization derived by EINSTEIN (1905) from his special theory of relativity, expressing the equivalence of mass and energy. Here, as in the equation of Einstein, we have a measure of energy E related approximately to a measure of mass m, through a factor of proportionality which is the square of a velocity of wave disturbances. Aside from noting an apparent pervasiveness of physico-mathematical forms in nature, one must be careful to observe that gravity wave phenomena, as here dealt with, require no departure from the mechanics of Newton for their treatment, so that no *identity* of meanings can be implied by the similarity. In this general connection see also STARR (1951).

It is perhaps of some concern to obtain instead of the inequality (8) the exact equation which results from the combination of (3) and (7). It may be written as

$$E = mc^2 - \varepsilon \qquad (9)$$

From what has previously been pointed out in regard to the magnitude of ε, its omission in (9) might cause an error (at most) of only a few per cent, and the use of the equality in (8) is correct up to but not including the fourth order of approximation.

People have asked me whether, disregarding the extensive incongruities present, there might be additional similarities between surface wave phenomena and relativity theory, besides those already cited. I wish therefore to mention one other, although its details are not altogether satisfactory. Let us consider all the fluid elements governed by the motions of a particular fully developed ideal wave train. Of all this infinity of particles none can have a velocity exceeding in magnitude the wave speed c. As a limiting case, only certain elements of measure zero can actually attain this critical speed, that is, those at the sharp crests of the waves. Furthermore, as fluid elements approach this state of motion, they also approach a state of infinite contraction along the direction of wave propagation. These facts are reminiscent of the Lorentz-Fitzgerald contraction. However, since the contraction must be accompanied by a transverse dilation, only an incomplete analogue of the Lorentz transformation can be present.

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