Passive scalar conditional statistics in a model of random advection

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We study numerically a model of random advection of a passive scalar by an incompressible velocity field of different prescribed statistics. Our focus is on the conditional statistics of the passive scalar and specifically on two conditional averages: the averages of the time derivative squared and the second time derivative of the scalar when its fluctuation is at a given value. We find that these two conditional averages can be quite well approximated by polynomials whose coefficients can be expressed in terms of scalar moments and correlations of the scalar with its time derivatives. With the fitted polynomials for the conditional averages, analytical forms for the probability density function (pdf) of the scalar are obtained. The variation of the coefficients with the parameters of the model result in a change in the pdf. Three different kinds of velocity statistics, (i) Gaussian, (ii) exponential, and (iii) triangular, are studied, and the same qualitative results are found demonstrating that the one-point statistics of the velocity field do not affect the statistical properties of the passive scalar.
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I. INTRODUCTION

In the study of fluid turbulence, one of the interesting problems is to understand the fluctuation statistics of velocity and temperature fields, and their derivatives. The statistics of velcocity derivatives or vorticity and temperature derivatives in turbulent flows, which are believed to be small-scale characteristics, have been known to deviate significantly from Gaussian and this is directly related to the problem of dissipative-range intermittency.[1] A series of experimental studies on Rayleigh–Bénard convection further reveal that the temperature fluctuation itself can also be non-Gaussian when the Rayleigh number is high enough.[2] Such a discovery has motivated several studies[3–9] to understand the statistics of a randomly advected passive scalar, which is a theoretically more tractable problem.

The statistics of any fluctuating quantity are described by its probability density function (pdf). It has been found that the pdf of any stationary fluctuation can be expressed in an exact formula in terms of two conditional averages, the averages of the time derivative squared and the second time derivative of the fluctuation for a given value of the fluctuation.[10,11] Suppose $X(t)$ is a physical quantity measured in a statistically stationary process. For simplicity, we take the mean and the standard deviation of $X(t)$ to be 0 and 1, respectively: $\langle X\rangle=0$ and $\langle X^2\rangle=1$. The angular bracket $\langle\cdots\rangle$ denotes the ensemble average. The pdf of X, $P(x)$, is then given by

$$P(x)=\frac{C_N}{\langle \dot{X}^2|X=x\rangle}\exp\left[\int_0^x \frac{\langle \ddot{X}|X=x'\rangle}{\langle \dot{X}^2|X=x'\rangle}dx'\right], \tag{1}$$

where an overdot indicates a time derivative and C_N is a constant fixed by normalization: $\int_{-\infty}^{\infty}P(x)dx=1$. The quantity $\langle \dot{X}^2|X=x\rangle$ denotes the average of the square of the time derivative of $X(t)$ when X is at a given value x. It is thus a conditional average and is generally a function of x. The conditional average of the second time derivative $\langle \ddot{X}|X=x\rangle$ is defined similarly. An analogous formula for statistically homogeneous fluctuations can also be derived.[11] Thus the problem of understanding the statistics of any fluctuation $X(t)$ is equivalent to the problem of understanding the two corresponding conditional averages $\langle \dot{X}^2|X=x\rangle$ and $\langle \ddot{X}|X=x\rangle$.

It has been found[12] that the following closed-form expression for the pdf:

$$P(x)=\frac{C_N}{\langle \dot{X}^2|X=x\rangle}\exp\left[\int_0^x -\frac{\langle \dot{X}^2\rangle x'}{\langle \dot{X}^2|X=x'\rangle}dx'\right], \tag{2}$$

is in good correspondence with measurements in high-Rayleigh-number and high-Reynolds-number flows when $X(t)$ is taken to be the turbulent temperature fluctuation $\delta T(t)$. Such an observation implies that the conditional average $\langle \ddot{\delta T}|\delta T\rangle$ is approximately $-\langle(\dot{\delta T})^2\rangle\delta T$.[10] Linearity for the conditional average of the second time derivative has also been found to hold approximately for spanwise vorticity data taken in several turbulent shear flows.[13] The existence of such simple general statistical feature in turbulence is quite surprising and understanding it remains a challenge. On the other hand, (2) does not work well for the time derivative of the temperature fluctuation $\partial[\delta T(t)]/\partial t$[12] which indicates that the function $\langle \partial^3(\delta T)/\partial t^3|\partial(\delta T)/\partial t\rangle$ deviates significantly from a linear function of $\partial(\delta T)/\partial t$.

Another major problem in turbulence is to understand the behavior of the structure functions of the velocity field or the scalar field. An n-th order structure function of a field is the ensemble average of the n-th power of the field difference separated by a certain distance. The specific question of interest is to study the scaling properties of these structure functions as a function of the separating distance. In particular, one is concerned with whether there is any deviation from the scaling predicted by Kolmogorov-type dimensional arguments, i.e., whether there is any anomalous scaling. Recently, it has been demonstrated[14] that the conditional average of the Laplacian of the field difference on the value of the field difference plays a crucial role in this scaling problem. Using the standard Taylor hypothesis to replace spatial gradients by time derivatives, such a conditional average is then of the form $\langle \ddot{X} | X=x \rangle$ (where X is the field difference) as discussed above.

In this paper, we report a numerical study of the conditional statistics of a passive scalar in a model of random advection. The model was first studied by one of the present authors and Tu[8] and will be described in more details in the next Section. It has been found that the passive scalar fluctuation changes from Gaussian to non-Gaussian upon variation of the parameters in this model. Such a change is similar to what was observed in experiments, as discussed in the beginning of this section. Thus, it is interesting to study how the two conditional averages behave and particularly how they change upon variation of the parameters. We find that the two conditional averages can be fitted quite well by polynomials whose coefficients can be expressed in terms of scalar moments and correlations of the scalar fluctuation with its time derivatives. The coefficients vary with the parameters of the model resulting in a change in the pdf of the passive scalar fluctuation. Another interesting issue is whether and how the statistical properties of the scalar field depend on those of the advecting velocity field. We have studied velocity fields of different prescribed statistics. The same qualitative results are found for the three different kinds of velocity statistics studied, showing that the one-point statistics of the velocity field do not affect the statistical properties of the passive scalar.

II. MODEL

We study the advection of a passive scalar $T(\mathbf{r},t)$, for example, the temperature field, by a random incompressible velocity field $\mathbf{u}(\mathbf{r},t)$. Such a process is described by the following equations:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T = \kappa \nabla^2 T, \tag{3a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3b}$$

where κ is the molecular diffusivity. Following an earlier study,[8] we do not solve the full advection-diffusion problem but resort to a simplified discrete model by evaluating (3) on a two-dimensional square lattice with a lattice spacing equals to ξ. Such coarse-graining has an effect of renormalizing the molecular transport coefficient thus we replace κ by an effective diffusivity D. The advecting velocity field is generated using the stream function $\phi(\mathbf{r},t)$ which is a scalar function along the third direction. To mimic a turbulent velocity field, we model $\phi(\mathbf{r},t)$ by a random field with prescribed statistics which has a correlation time τ. The random field is generated independently at each lattice site such that the stream function and thus the velocity field has a correlation length equal to the lattice spacing ξ. The noise strength of the stream function is measured by its standard deviation ϕ_0. The typical size of velocity fluctuation is given by $u_0 \equiv \phi_0/\xi$.

Using τ, ξ, and u_0 as typical time, length, and velocity scales, we nondimensionalize the equation of motion. There are three time scales in the problem: the velocity correlation time τ, the advection time ξ/u_0, and the diffusion time ξ^2/D, giving rise to two independent dimensionless parameters. The resulting equation is

$$\frac{\partial T(i,j)}{\partial t} + K\mathbf{u}(i,j) \cdot \nabla_{ij} T = \frac{1}{C} \nabla_{ij}^2 T(i,j), \tag{4}$$

with the two dimensionless parameters being K and C. The parameter K is the ratio of the velocity correlation time to the advection time, $u_0\tau/\xi$, and is known as the Kubo number. The parameter C is the ratio of the diffusion time to the velocity correlation time: $\xi^2/(D\tau)$. The product of K and C gives a Péclet-like number $u_0\xi/D$.

We use finite difference method with a small time step Δt to integrate (4) in time. The system size is $N \times N$ and $N=31$ is used in the present work. A relatively small size is sufficient as we shall evaluate the statistics by averaging over time. The random stream function ϕ at each lattice site is updated every m time steps so that $\tau = m\Delta t$. The boundary condition for both the velocity and the scalar fields is periodic in the i-direction. In the j-direction, the velocity field is no-slip on both the top and bottom boundaries: $\mathbf{u}(i,j=1) = \mathbf{u}(i,j=N) = 0$ while the scalar field satisfies a fixed-difference condition: $T(i,j=1)=0$ and $T(i,j=N)=1$. To study the scalar statistics, we measure T at the center of the system, $[(N+1)/2, (N+1)/2]$, as a function of time after the system reaches the steady state. A long time series with at least 4×10^6 data points are used to get good statistics. Different probability distributions are prescribed for the stream function to generate velocity field with three different kinds of statistics: (i) Gaussian, (ii) exponential, and (iii) triangular. For all the three cases, we fix the parameter K at 1 such that the velocity correlation time is the same as the advection time and study the conditional statistics of T as a function of the parameter C.

III. RESULTS

A. Gaussian velocity statistics

We first consider the case of velocity field having Gaussian statistics. Such a velocity field is generated by a stream function which has a Gaussian distribution. The parameter C is varied from 0.01 to 13.3 by varying D, ϕ_0, and τ. As discussed in Ref. 8, the mean scalar profile is almost linear in j, with a gradient of $1/N$, for all the values of C studied. The mean profiles are not exactly linear and have a slight dependence on i, with the spread in i slightly larger for larger C. For each value of C, we use the time series measured at the center of the system to calculate the pdf, $P(X=x)$, of the standardized scalar fluctuations $X(t)$ which is defined by

$$X \equiv \frac{T - \langle T \rangle}{\sqrt{\langle (T - \langle T \rangle)^2 \rangle}}, \tag{5}$$

and the two conditional averages,

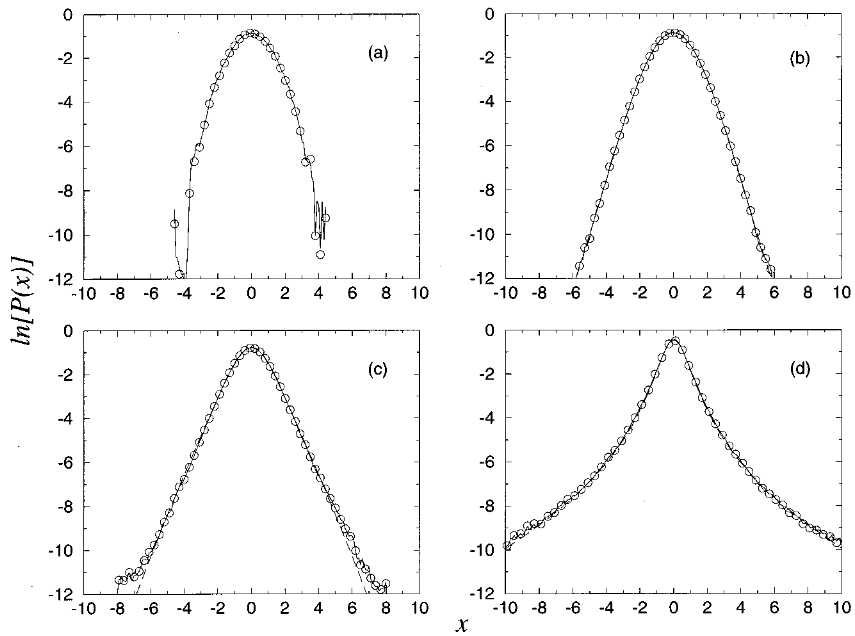

FIG. 1. The probability density function $P(x)$ of normalized scalar fluctuation evaluated by various methods for the case of Gaussian velocity statistics. Directly from the data (solid lines); using Eq. (1) with the conditional averages $q(x)$ and $r(x)$ from data (circles); using analytical forms Eqs. (13) and (16) resulted from the polynomial fits of $q(x)$ and $r(x)$ (dashed lines). Good agreement among the three can be seen. (a) $C=0.01$, (b) $C=3.00$, (c) $C=6.67$, and (d) $C=13.3$.

$$q(x) \equiv \frac{\langle \dot{X}^2 | X=x \rangle}{\langle \dot{X}^2 \rangle}, \tag{6}$$

$$r(x) \equiv \frac{\langle \ddot{X} | X=x \rangle}{\langle \dot{X}^2 \rangle}. \tag{7}$$

The averages are evaluated by averaging over time.

In Fig. 1, we plot $P(x)$ (solid line) for the various values of C studied. As reported in Ref. 8, the pdf varies from a Gaussian to distribution with flatter-than-Gaussian tails as C increases (or as the Péclet-like number increases since K is fixed to be 1). Using the calculated $q(x)$ and $r(x)$, we evaluate the right hand side of (1) and display the result as circles in the same figure. It is clear that the circles coincide very well with the solid line verifying that the formula (1) holds well for all the pdf's regardless of whether they are Gaussian or not.

In Fig. 2, we show the conditional averages $q(x)$ and $r(x)$ as a function of C. For small C, $q(x)$ is approximately independent of x and its value is about 1 [which is what one would get by definition if $q(x)$ is exactly a constant]. For larger values of C, $q(x)$ becomes a concave, quadratic-like, function in x with the functional dependence on x becoming stronger and stronger as C further increases. On the other hand, the conditional average $r(x)$ is almost a linear function, $-x$, for both small and large values of C. For intermediate values of C, the deviation of $r(x)$ from $-x$ cannot be neglected and $r(x)$ is approximated better by a cubic function in x.

To quantify the variation of $q(x)$ and $r(x)$ as a function of C, we fit them by polynomials in x. Because of the T to $-T$ symmetry at the center of the system, $P(x)$ is symmetric

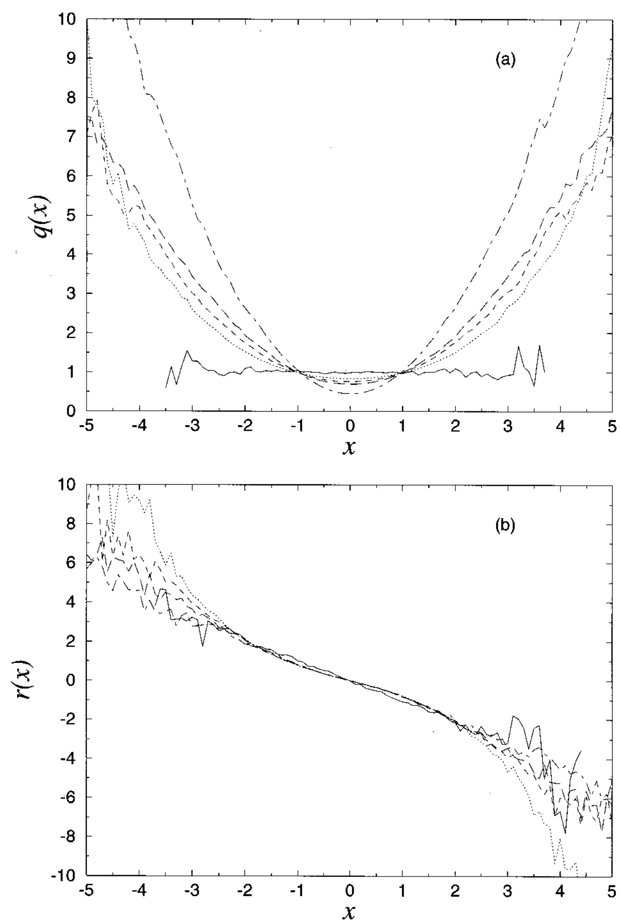

FIG. 2. The conditional averages (a) $q(x)$ and (b) $r(x)$ as a function of the parameter C for the case of Gaussian velocity statistics. $C=0.01$ (solid line), $C=3.00$ (dotted line), $C=6.67$ (dashed line), $C=9.00$ (long dashed line), and $C=13.3$ (dot–dashed line).

TABLE I. Fitted values of the coefficients b, A_0, A_2, A_4, c, e, α, β, α', and β' (see the text) in the case of Gaussian velocity statistics.

C	b	A_0	A_2	A_4	c	e	α	β	α'	β'
0.01	0.002	—	—	—	0.03	0.009	—	—	—	—
0.20	0.07	—	—	—	0.19	0.08	0.21	0.5	—	—
0.80	0.12	—	—	—	0.32	0.15	0.79	0.42	—	—
1.00	0.13	—	—	—	0.32	0.15	0.98	0.39	—	—
3.00	—	0.83	0.15	0.006	0.33	0.14	—	—	5.29	5.16
5.00	—	0.79	0.21	0.002	0.25	0.09	—	—	9.66	7.38
6.67	0.26	—	—	—	0.24	0.07	2.17	0.11	—	—
9.00	0.30	—	—	—	0.07	0.02	2.51	0.03	—	—
11.0	0.38	—	—	—	0.02	0.02	2.27	0.02	—	—
13.3	0.54	—	—	—	0.09	0.009	1.84	0.007	—	—

as observed in Fig. 1. This symmetry implies that $q(x)$ should be an even function of x while $r(x)$ should be an odd function of x. Thus, we fit $q(x)$ by an even quadratic polynomial: $a+bx^2$. By definition, $\int_{-\infty}^{\infty} P(x)q(x)dx=1$ and this constrains b to be $1-a$ so there is only one fitting parameter which we choose as b. That is, the fitting form for $q(x)$ is

$$q_{fit}(x)=1-b(1-x^2). \tag{8}$$

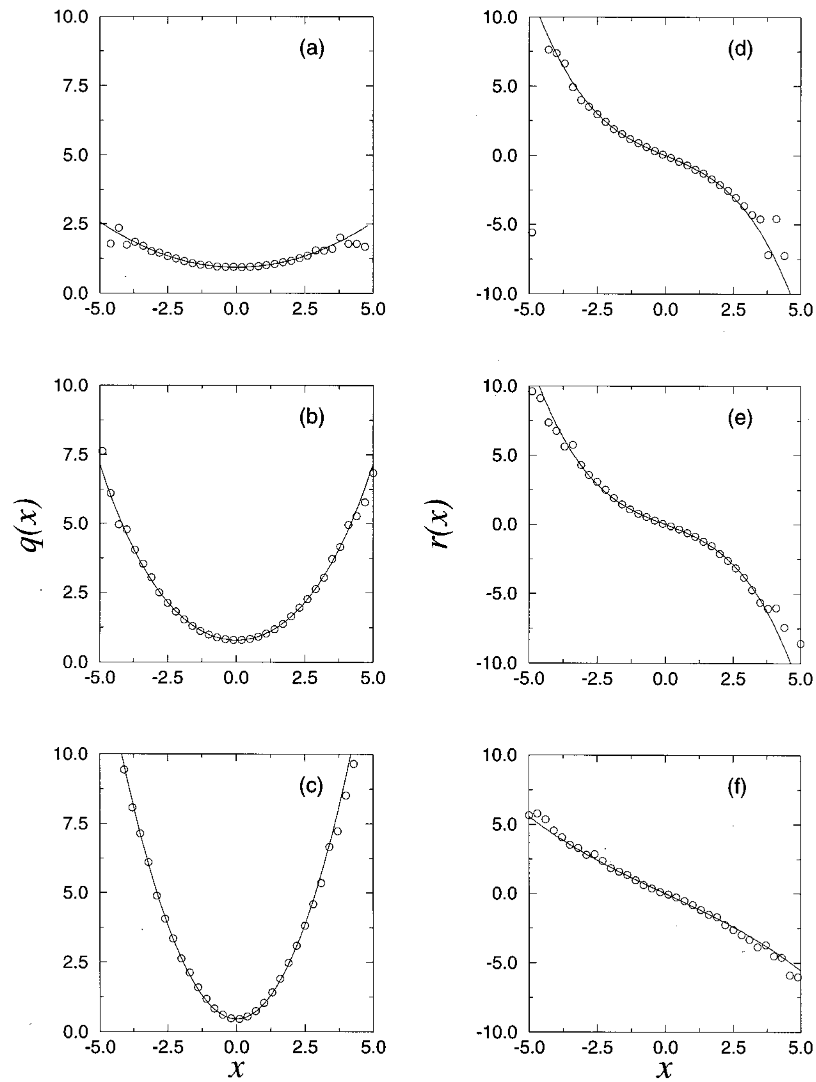

FIG. 3. Some representative polynomial fits of $q(x)$ and $r(x)$ for the case of Gaussian velocity statistics. (a) and (d) $C=0.20$, (b) and (e) $C=5.00$, (c) and (f) $C=13.3$. Circles are data points while solid lines are polynomial fits.

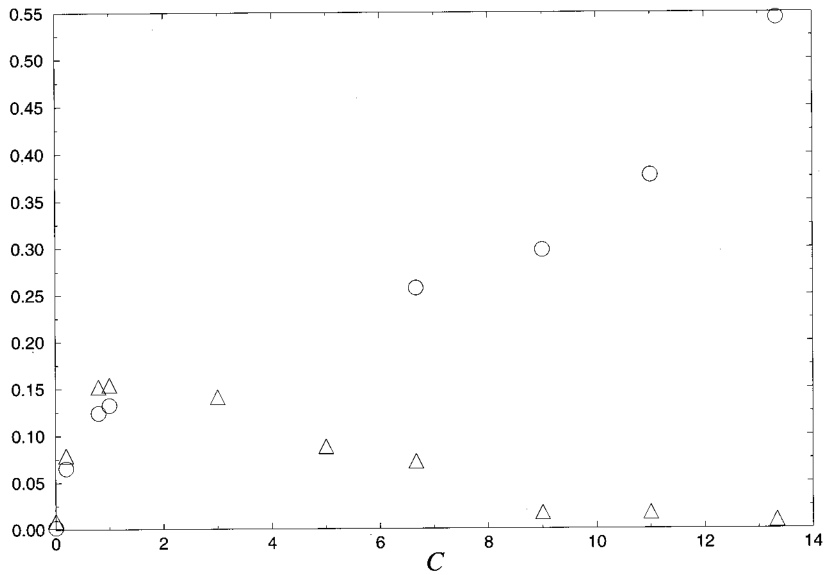

FIG. 4. The parameters b (cirles) and e (triangles) as a function of C for the case of Gaussian velocity statistics.

We find that this form works generally well except for the cases $C=3$ and 5 for which we need a quartic polynomial:

$$\tilde{q}_{fit}(x)=A_0+A_2x^2+A_4x^4. \tag{9}$$

For $r(x)$, we use an odd cubic polynomial:

$$r_{fit}(x)=-(1-c)x-dx^3, \tag{10}$$

as a fitting form. The stationarity condition gives $\langle \ddot{X}X\rangle = -\langle \dot{X}^2\rangle$ which constrains d to be $c/\langle X^4\rangle$. With $\langle X^4\rangle$ evaluated from data, there is again only one fitting parameter c. However, for $C=9$,11.0, and 13.3, the pdf of the fluctuation has very flat tails which makes an accurate evaluation of $\langle X^4\rangle$ from the measurements difficult. Thus in these cases, we treat d as an independent parameter in the fit. In Table I, we display the fitted values of the coefficients. Using these fitted values, we construct q_{fit} or $\tilde{q}_{fit}$ and r_{fit} which are found to agree well with q and r. Some representative fits are shown in Fig. 3.

We now focus on the cases where (8) and (10) are good fits for $q(x)$ and $r(x)$. The parameter b measures how much $q(x)$ deviates from a constant $(=1)$. It has been shown that if $r(x)$ is a linear function of x then $r(x)$ has to be $-x$.[10] Thus, the parameter c can be taken as a measure of how much $r(x)$ deviates from linearity with the importance of the nonlinear term in $r(x)$ given directly by $e\equiv d/(1-c)$. In Fig. 4, we plot b and e as a function of C. As C increases, b in-

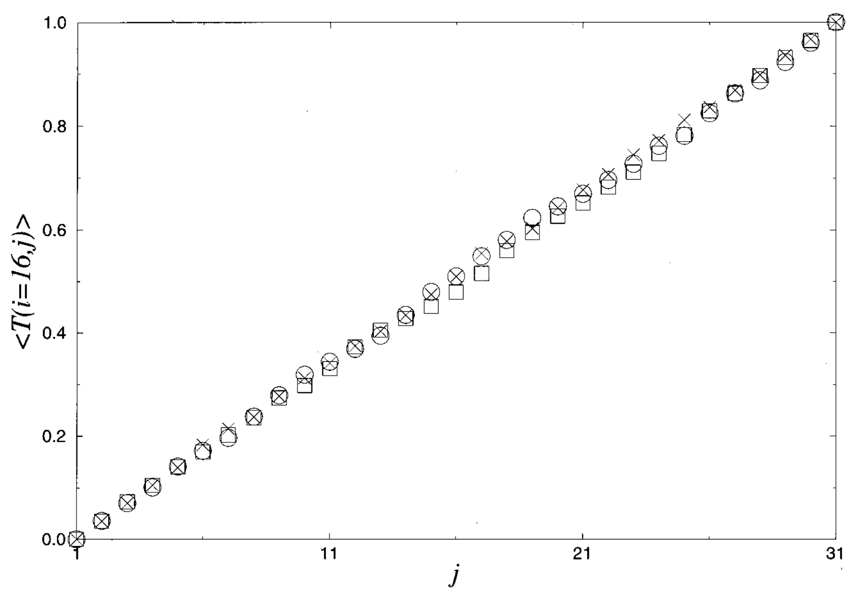

FIG. 5. A comparison of the mean scalar profile along the centerline of the lattice, at $i=16$, for the three different kinds of velocity statistics studied: Gaussian (circles), exponential (squares), and triangular (crosses). The parameter C is 13.3. It can be seen that the mean profile is almost linear regardless of what the velocity statistics are.

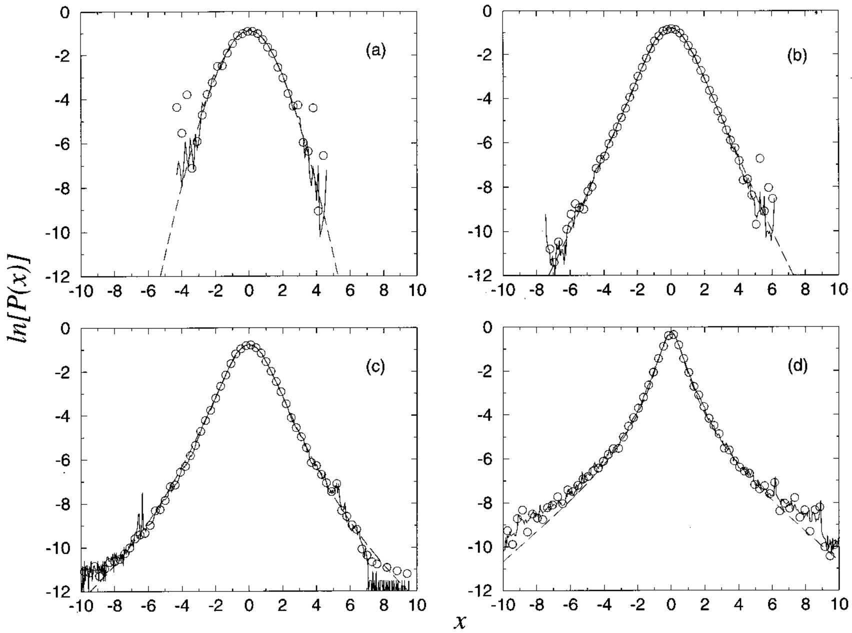

FIG. 6. Similar to Fig. 1 for the case of exponential velocity statistics. (a) $C=0.01$, (b) $C=1.00$, (c) $C=5.00$, and (d) $C=13.3$.

creases montonically while e first increases then decreases as C further increases. Suppose (8) is taken as an exact form for $q(x)$, then the coefficient b can be expressed explicitly in terms of moments of X and correlations of X and its time derivatives:[15]

$$b=\frac{\langle \dot{X}^2 X^{2n}\rangle/\langle \dot{X}^2\rangle-\langle X^{2n}\rangle}{\langle X^{2n+2}\rangle-\langle X^{2n}\rangle}, \quad \text{for any integer } n. \tag{11}$$

Similarly, if we take (10) as an exact form for $r(x)$, then c [with $e=c/[(1-c)\langle X^4\rangle]$] can be evaluated to be

$$c=\frac{\langle X^{2n+2}\rangle-(2n+1)\langle \dot{X}^2 X^{2n}\rangle/\langle \dot{X}^2\rangle}{\langle X^{2n+2}\rangle-\langle X^{2n+4}\rangle/\langle X^4\rangle}, \quad \text{for any integer } n. \tag{12}$$

In (12), the relation $\langle \ddot{X} X^{2n+1}\rangle=-(2n+1)\langle \dot{X}^2 X^{2n}\rangle$ is used, which is valid for stationary fluctuation X. The parameter b is proportional to the difference between $\langle \dot{X}^2 X^{2n}\rangle$ and $\langle X^{2n}\rangle\langle \dot{X}^2\rangle$. Thus, how strong $\dot{X}^2$ and X^{2n} are statistically

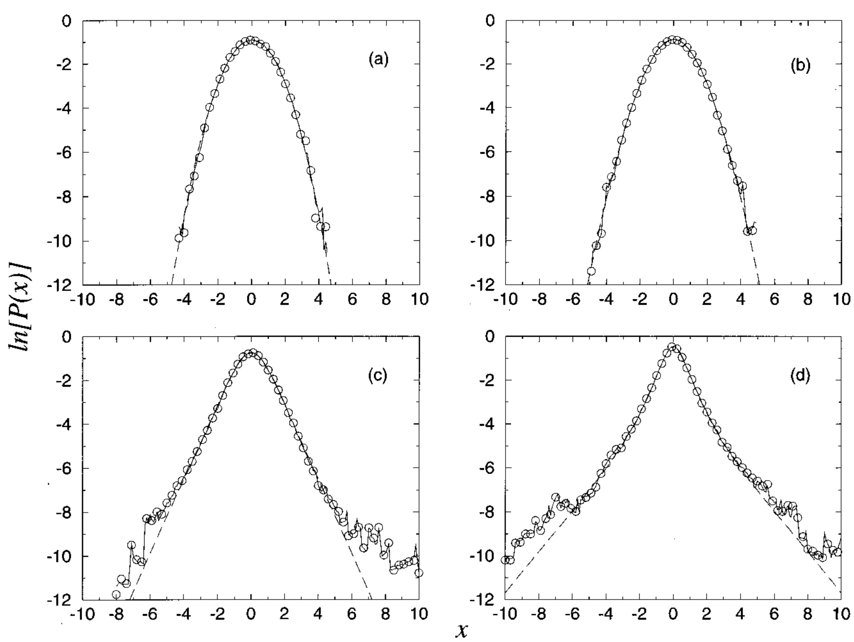

FIG. 7. Similar to Fig. 1 for the case of triangular velocity statistics. (a) $C=0.05$, (b) $C=3.00$, (c) $C=9.00$, and (d) $C=13.3$.

correlated with each other controls how much $q(x)$ deviates from a constant. From (12), the parameter c deviates from zero when $\langle X^{2n+2}\rangle$ deviates from $(2n+1)\langle \dot{X}^2 X^{2n}\rangle/\langle \dot{X}^2\rangle$.

When the pdf of $X(t)$ is Gaussian, we have $\langle X^{2n+2}\rangle=(2n+1)\langle X^{2n}\rangle$. Thus the uncorrelation of $\dot{X}^2$ and X^{2n} is equivalent to $c=0$ or, in other words, $q(x)$ being 1 and $r(x)$ being $-x$ are equivalent in this case. From (1), if $q(x)$ is algebraic then it has to be 1 for a Gaussian $P(x)$. Hence for a Gaussian $P(x)$, we have $q(x)=1$ and $r(x) =-x$ unless q and r are nonalgebraic. For the case that not only $P(x)$ is Gaussian but that $X(t)$ is a Gaussian process, it can be proved exactly that $q(x)=1$ and $r(x)=-x$ as discussed in Ref. 10. On the other hand, for non-Gaussian $P(x)$, the linearity of $r(x)$, if observed, is a nontrivial result in the sense discussed above.

Using (8) and (10) in (1), we obtain an analytical expression for $P(x)$:

$$P(x)=\frac{C_N}{(1-b+bx^2)^{\alpha}}\exp(-\beta x^2), \tag{13}$$

where

$$\alpha=1+\frac{1-c}{2b}-\frac{d(1-b)}{2b^2}, \tag{14}$$

$$\beta=\frac{d}{2b}. \tag{15}$$

We note that if α is small, (13) is a Gaussian slightly modified by an algebraic factor. On the other hand, if β is small then (13) is of the form of a Lorentzian raised to α so that $P(x)$ has algebraic tails. For the cases $C=3$ and $C=5$, $q(x)$ is better fitted by (9) and the corresponding analytic expression for $P(x)$ is given by

$$P(x)=\frac{C_N}{(A_0+A_2x+A_4x^4)^{\alpha'}}\left[\frac{2A_0+(A_2+\Delta)x^2}{2A_0+(A_2-\Delta)x^2}\right]^{\beta'}, \tag{16}$$

with

$$\alpha'=1+\frac{d}{4A_4}, \tag{17}$$

$$\beta'=-\frac{1}{2\Delta}\left[(1-e)-\frac{A_2 d}{2A_4}\right], \tag{18}$$

$$\Delta=\sqrt{A_2^2-4A_0A_4}. \tag{19}$$

[Note that $A_2^2>4A_0A_4$ (see Table I) so that Δ is real.] The asymptotic behavior of $P(x)$ as $|x|\to\infty$ is given by $x^{-4\alpha'}$. We note that the value of α' is quite large in these two cases such that $P(x)$ has very fast-decreasing tails which are practically indistinguishable from exponential tails. The values of α and β, or α' and β', for the different values of C (except for $C=0.01$[16]) are shown in Table I. We calculate (13) and (16) and compare them (dashed lines) to the directly calculated pdf's (solid lines) in Fig. 1. The good agreement is a confirmation that (8) or (9) and (10) are good approximations for the conditional averages q and r.

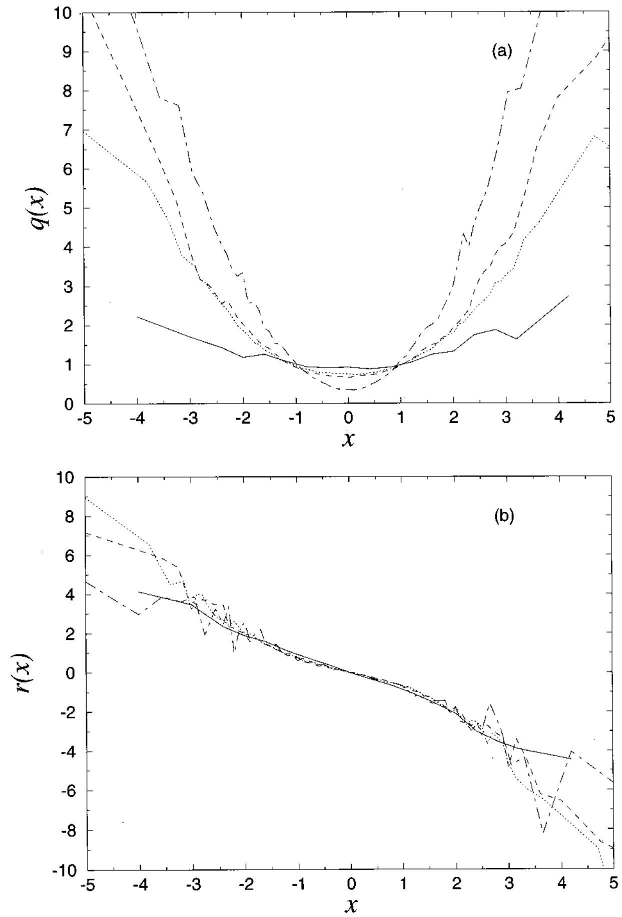

FIG. 8. Similar to Fig. 2 for the case of exponental velocity statistics. $C=0.01$ (solid line), $C=1.00$ (dotted line), $C=5.00$ (dashed line), and $C=13.3$ (dot–dashed line).

B. Other velocity statistics

Besides considering velocity field that has a Gaussian distribution, we have studied two other types of velocity statistics. The first is an exponentially distributed velocity field which is generated by using a stream function whose distribution is given by the modified Bessel function $K_0(|\phi|/\phi_0)/(\pi\phi_0)$. For the second one, we use a uniformly distributed stream function which produces a velocity field that has a triangular distribution.

The mean scalar profiles are again found to be almost linear in j. A comparison of the mean profile along the centerline, at $i=16$, is made for the three kinds of velocity statistics and is shown in Fig. 5. We see that the mean profile is not affected much by the velocity statistics.

We analyze the scalar measurements as in the case of Gaussian velocity statistics. First, we find that the pdf of the scalar fluctuation changes from Gaussian to non-Gaussian with flatter tails as observed before. The pdf's are shown in Figs. 6 and 7, respectively, for exponential and triangular velocity statistics. Equation (1) is again verified. Second, we study the variation of the conditional averages q and r as a function of C. Results for exponential velocity statistics are shown in Fig. 8 while those for triangular statistics are

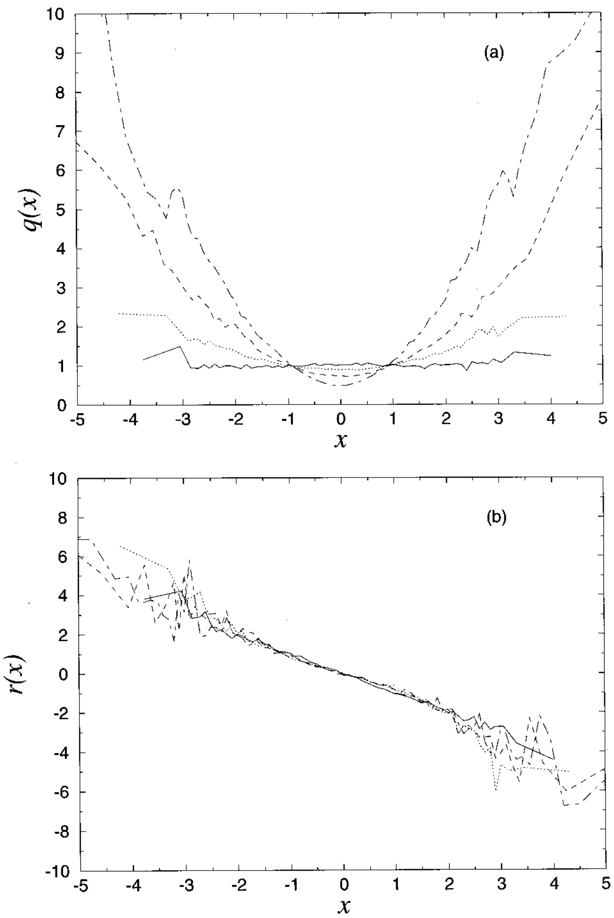

FIG. 9. Similar to Fig. 2 for the case of triangular velocity statistics. $C=0.05$ (solid line), $C=3.00$ (dotted line), $C=9.00$ (dashed line), and $C=13.3$ (dot–dashed line).

shown in Fig. 9. Comparing them with Fig. 2, we find similar changes as C increases.

As before, we fit q and r using polynomials. We find that (8) and (10) are good fits in general with the exception that (9) is a better fit for q for intermediate values of C ($C=1.00, 3.00$, and 5.00) in the case of exponential velocity statistics. The parameter d has to be treated as an independent parameter in the fit r_{fit} for $C\geqslant 9$ as discussed before. The values of the fitted coefficients are displayed in Table II and Table III, respectively, for the two types of velocity statistics. As C increases, the parameter b increases while the parameter e first increases then decreases as observed before.

TABLE III. Fitted values of the coefficients b, c, e, α, and β (see the text) in the case of triangular velocity statistics.

C	b	c	e	α	β
0.05	0.008	0.02	0.007	12.45	0.40
3.00	0.11	0.23	0.09	1.95	0.32
9.00	0.26	0.18	0.05	2.33	0.08
13.3	0.54	0.23	0.04	1.68	0.03

Following the discussion in the previous subsection, using (8) or (9) and (10) in (1) lead to analytical result for the pdf given by (13) or (16). The values for α and β, or α' and β', are also shown in Tables II and III for the two types of statistics. Comparisons of the analytic forms (dashed lines) with the directly calculated pdf's (solid lines) are done in Figs. 6 and 7. Good agreement is again found.

IV. DISCUSSION AND SUMMARY

We have studied numerically a model of a passive scalar advected by a random incompressible velocity field in two dimensions. As the parameter C in the model increases, the pdf changes from Gaussian to distribution with flatter-than-Gaussian tails. This change is qualitatively the same for the three different velocity statistics studied: (i) Gaussian, (ii) exponential, and (iii) triangular. As expected, (1) works for the pdf of the scalar fluctuation so that the pdf is expressed in terms of the conditional averages q and r. Thus, the change in the pdf is reflected by changes in q and r.

The changes in q and r can be quantified by fitting them with polynomial forms and then studying the behavior of the fitted coefficients as a function of C. The conditional average $q(x)$ can generally be well fitted by an even quadratic polynomial except for intermediate values of C for which a quartic polynomial is a better fit. In general, $q(x)$ deviates more and more from a constant as C increases. On the other hand, the conditional average $r(x)$ can be well fitted by an odd cubic polynomial for all the values of C studied. We find that r is close to a linear function for both small and large values of C while for intermediate values of C, the cubic term cannot be neglected. The approximate linearity of $r(x)$ when the pdf is non-Gaussian is a nontrivial result which has yet to be understood.

With the form of the fitted polynomials for q and r, we obtain analytical results for the pdf using (1). We find that for small values of C, the pdf is a Gaussian, slightly modified

TABLE II. Fitted values of the coefficients b, A_0, A_2, A_4, c, e, α, β, α', and β' (see the text) in the case of exponential velocity statistics.

C	b	A_0	A_2	A_4	c	e	α	β	α'	β'
0.01	0.10	—	—	—	0.15	0.05	3.08	0.25	—	—
0.05	0.16	—	—	—	0.19	0.07	2.66	0.16	—	—
1.00	—	0.71	0.27	0.005	0.31	0.11	—	—	4.43	2.40
3.00	—	0.67	0.31	0.006	0.37	0.13	—	—	4.39	2.59
5.00	—	0.69	0.26	0.01	0.30	0.08	—	—	1.97	−0.57
9.00	0.43	—	—	—	0.19	0.05	1.89	0.05	—	—
11.0	0.45	—	—	—	0.25	0.05	1.79	0.04	—	—
13.3	0.68	—	—	—	0.16	0.03	1.60	0.02	—	—

by an algebraic factor. For intermediate values of C, the pdf has power-law tails that decay very fast and are thus indistinguishable from exponential tails. For large values of C, the pdf is of the form of a generalized Lorentzian which has algebraic tails.

Results for the three different types of velocity statistics studied are found to be qualitatively the same. This demonstrates that the one-point statistics of the velocity field do not play a role in determining the pdf and the conditional averages of the passive scalar. The relevant parameter is C which is a measure of the relative size of the effective diffusion time to the correlation time of the velocity field.

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[1] See, for example, M. Nelkin, ''Universality and scaling in fully developed turbulence,'' Adv. Phys. **43**, 143 (1994), and references therein.

[2] F. Heslot, B. Castaing, and A. Libchaber, ''Transitions to turbulence in helium gas,'' Phys. Rev. A **36**, 5870 (1987); M. Sano, X.-Z Wu, and A. Libchaber, ''Turbulence in helium−gas free convection,'' Phys. Rev. A **40**, 6421 (1989).

[3] Y. G. Sinai and V. Yakhot, ''Limiting probability distributions of a passive scalar in a random velocity field,'' Phys. Rev. Lett. **63**, 1962 (1989).

[4] H. Chen, S. Chen, and R. H. Kraichnan, ''Probability distributions of a stochastically advected scalar fields,'' Phys. Rev. Lett. **63**, 2657 (1989).

[5] A. Pumir, B. I. Shraiman, and E. D. Siggia, ''Exponential tails and random advection,'' Phys. Rev. Lett. **66**, 2984 (1991).

[6] R. Bhagavatula and C. Jayaprakash, ''Non-Gaussian distributions in extended dynamical systems,'' Phys. Rev. Lett. **71**, 3657 (1993).

[7] M. Holzer and A. Pumir, ''Simple models of non-Gaussian statistics for a turbulently advected passive scalar,'' Phys. Rev. E **47**, 202 (1993).

[8] E. S. C. Ching and Y. Tu, ''Passive scalar fluctuations with and without a mean gradient: A numerical study,'' Phys. Rev. E **49**, 1278 (1994).

[9] M. Chertkov, G. Falkvoich, I. Kolokolov, and V. Lebedev, ''Statistics of a passive scalar advected by a large-scale two-dimensional velocity field: Analytic solution,'' Phys. Rev. E **51**, 5609 (1995).

[10] S. B. Pope and E. S. C. Ching, ''Stationary probability density functions: An exact result,'' Phys. Fluids A **5**, 1529 (1993).

[11] E. S. C. Ching, ''General formula for stationary or statistically homogeneous probability density functions,'' Phys. Rev. E **53**, 5899 (1996).

[12] E. S. C. Ching, ''Probability densities of turbulent fluctuations,'' Phys. Rev. Lett. **72**, 283 (1993).

[13] J. Mi and R. A. Antonia, ''General relation for stationary probability density functions,'' Phys. Rev. E **51**, 4466 (1995).

[14] E. S. C. Ching, V. L'vov, and I. Procaccia, ''Fusion rules and conditional statistics in turbulent advection,'' Phys. Rev. E **54**, R4520 (1996); E. S. C. Ching, V. L'vov, E. Podivilov, and I. Procaccia, ''Conditional statistics in scalar turbulence: Theory versus experiment,'' Phys. Rev. E **54**, 6364 (1996).

[15] B. Castaing, E. S. C. Ching, D. Lohse, A. Naert, G. Stolovitzky, and J. Werne, ''Conditional statistics in the analysis of turbulent data,'' preprint (1996).

[16] In the case of $C=0.01$, b is very small. As a result, the values of α and β depend very sensitively on the estimated value of d, rendering them inaccurate. Thus these estimates are omitted from Table I.