**Intermittency and Multifractality in Two-Dimensional Turbulence with Drag**

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**Turbulence: 2-D versus 3-D**

**3-D Turbulence**
- Navier-Stokes momentum equation
  \[
  \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V} + \vec{f} \quad (\nabla \cdot \vec{V} = 0)
  \]
- Kolmogorov’s Phenomenology \((r < 4/3)\)
  - mean energy dissipation rate \(\epsilon\)
  - structure functions
    \[
    \left\langle |\vec{V}(x) - \vec{V}(y)|^q \right\rangle \sim r^{\alpha_q} \quad \alpha_q = \frac{D_q - 1}{q - 1}
    \]
- experiments contradict Kolmogorov’s hypotheses

**2-D Turbulence**
- Vorticity equation \((\omega = \vec{V} \cdot \nabla \vec{V})\)
  \[
  \frac{\partial \vec{\omega}}{\partial t} + \vec{\omega} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{\omega} + \vec{f} \quad (\nabla \cdot \vec{\omega} = 0)
  \]
- Kraichnan’s Phenomenology (for forward cascade)
  - mean enstrophy dissipation rate \(\epsilon\)
  - structure functions
    \[
    \left\langle |\vec{\omega}(x) - \vec{\omega}(y)|^q \right\rangle \sim r^{\alpha_q} \quad \alpha_q = \frac{D_q - 1}{q - 1}
    \]
- recent experiment (magnetically forced stratified flow) supports Kraichnan’s theory

**Why care about 2-D Turbulence?**

- **Theory for \(\zeta_0\)**
  - relate \(\zeta_0\) to the finite-time Lyapunov exponent \(k\)
    \[
    k(\vec{x}, t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \ln \left| \frac{dG(\vec{x})}{dG(\vec{x} + \epsilon \delta \vec{x})} \right|
    \]
  - probability density function of \(k\), \(P(h, t)\) with large time asymptotic form
    \[
    P(h, t) \sim \exp[-|G(h)|]
    \]
  - theoretical result for \(\zeta_0\)
    \[
    \zeta_0 = \min \left\{ \frac{G(h)}{h} : \frac{dG(h)}{dh} \right\}
    \]

**Quantifying Multifractality**

Divide the region \(R\) occupied by the fluid into grid of square boxes \(R_\epsilon(\alpha)\) of size \(\epsilon\)

- Measure \(p(\epsilon)\) \(\equiv \frac{\int_{R_\epsilon(\alpha)} |\vec{\omega}(\vec{x})|^q d\vec{x}}{\int_{\Omega} |\vec{\omega}(\vec{x})|^q d\vec{x}}\)

Generalized Dimension \(D_q\)

- Singularity Spectrum \(f(\alpha)\): \(p(\epsilon) \sim \epsilon^{-\alpha} ; \quad N(\alpha) \sim \epsilon^{-f(\alpha)}\)

- If the measure \(p(\epsilon)\) is multifractal, then
  - \(D_q\) varies with \(\alpha\)
  - \(f(\alpha)\) is a non-trivial function of \(\alpha\)

**Relation between \(D_q\) and \(\zeta_0\)**

At the dissipative scale \(r_d\), the vorticity field is smooth out by the action of viscosity, thus

\[
\int_{R_\epsilon(\alpha)} |\vec{\omega}(\vec{x})|^q d\vec{x} \sim r_d^q |\vec{\omega}(\vec{x})|^q
\]

It then follows,

\[
D_q = 2 - \frac{\zeta_0}{q - 1}
\]

**Conclusion**

For two-dimensional turbulence with linear drag,
- the vorticity field is intermittent, \(\zeta_0 \neq \zeta_3\)
- \(\zeta_0\) can be obtained in terms of the finite-time Lyapunov exponent and the drag coefficient \(\mu\)
- the measure based on the viscous enstrophy dissipation is multifractal, \(D_q \neq constant\)
- \(D_q\) is related to \(\zeta_0\)